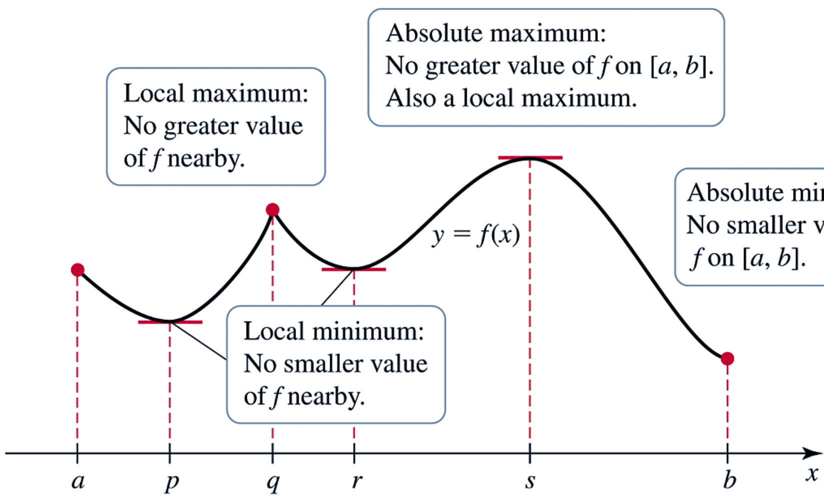
Calculus H Name _____________

Notes Ch. 4 (1,2)

4.1 Maxima and Minima

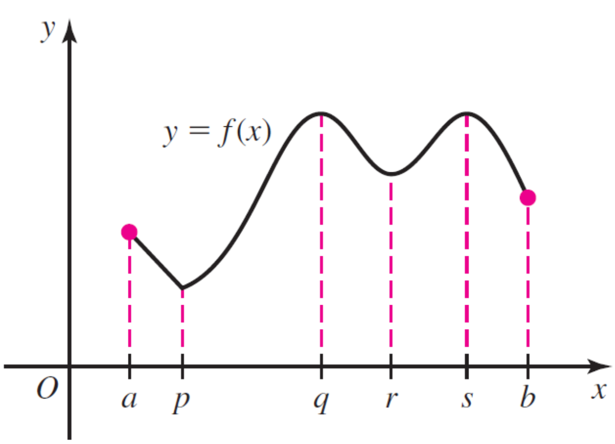
With a working understanding of derivatives, we now undertake one of the fundamental tasks of calculus: analyzing the behavior of functions and producing accurate graphs of them.



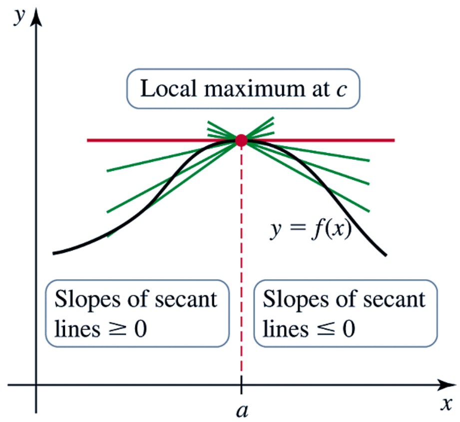
DEFINITION: Local Maximum and Minimum Values

Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a local maximum value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a local minimum value of f.

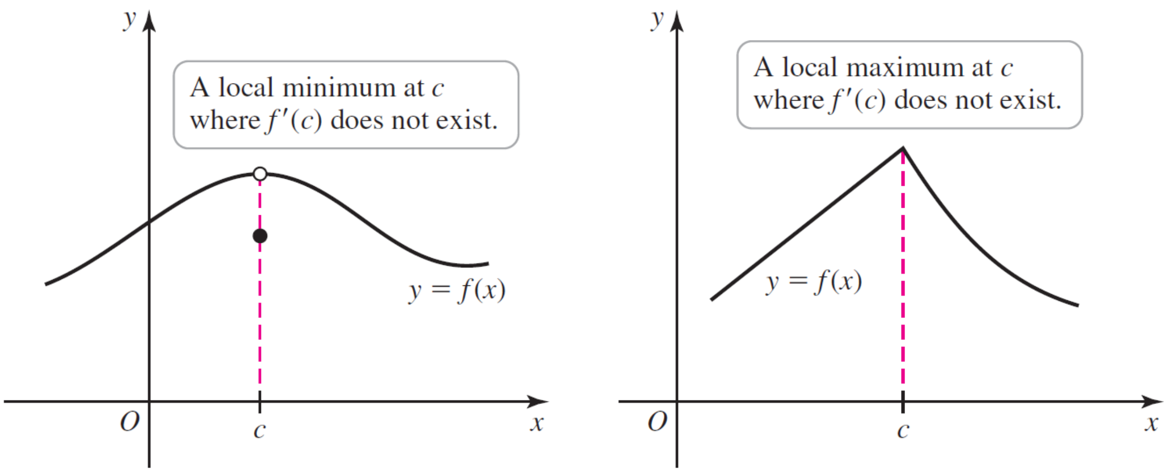
EXAMPLE: The figure below shows the graph of a function defined on $[a, b]$. Identify the location of the various maxima and minima using the terms absolute and local.



THEOREM 4.2 Local Extreme Value Theorem
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

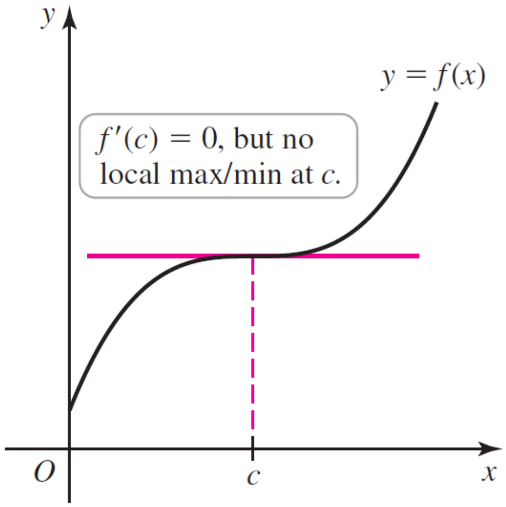


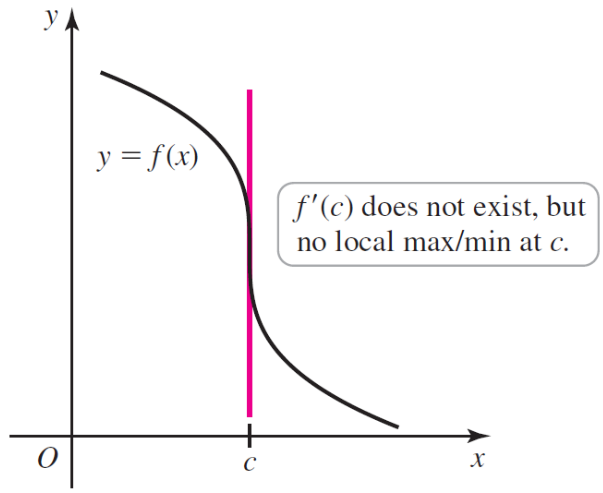
Local extrema can also occur at points c where $f'(c)$ does not exist. The figure below shows two such cases, one in which c is a point of discontinuity and one in which f has a corner point at c. Because local extrema may occur at points c where $f'(c) = 0$ or where $f'(c)$ does not exist, we make the following definition.



DEFINITION: Critical Point
An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a critical point of f.

The converse to Theorem 4.2 is not always true. Just because $f'(c) = 0$ doesn't mean there has to be a local maximum or minimum. It is also possible that $f'(c)$ fails to exist , with no local extreme value occurring at c. Critical points **are candidates** for the location of local extreme values.





EXAMPLE: Find the critical points of $f(x) = x^2 \ln x$.

Find the critical points of the following functions.

1. $f(x) = 3x^2 - 4x + 2$

2. $f(x) = \frac{1}{8}x^3 - \frac{1}{2}x$

3. $f(x) = \frac{x^3}{3} - 9x$

4. $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10$

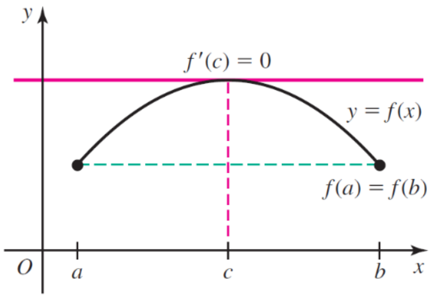
5. $f(x) = 3x^3 + \frac{3x^2}{2} - 2x$

6. $f(x) = \frac{4x^5}{5} - 3x^3 + 5$

4.2 Rolle's Theorem

Rolle's Theorem

Consider a function f that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Furthermore, assume f has the special property that $f(a) = f(b)$ (see figure below). The statement of Rolle's Theorem is not surprising: It says that somewhere between a and b, there is at least one point at which f has a horizontal tangent line (critical point).



THEOREM 4.3 Rolle's Theorem
Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.

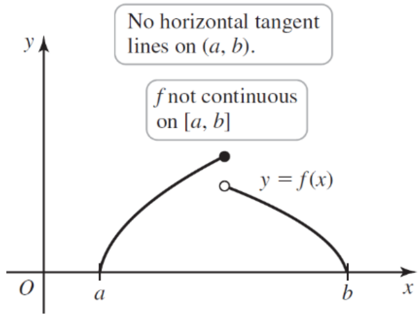
Some consequences of this:

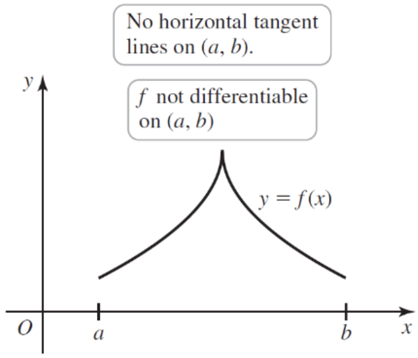
- Case I: $f(x) = k$, for some constant k.
  Derivative is 0 everywhere so pick any value in (a, b).

- Case II: $f(x) > f(a)$ for some x in (a, b).
  There must be a maximum value somewhere on (a, b).

- Case III: $f(x) < f(a)$ for some x in (a, b).
  There must be a minimum value somewhere on (a, b).

Why does Rolle's Theorem require continuity? A function that is not continuous on $[a, b]$ may have identical values at both endpoints and still not have a horizontal tangent line at any point on the interval. Similarly, a function that is continuous on $[a, b]$ but not differentiable at a point of (a, b) may also fail to have a horizontal tangent line.





EXAMPLE: Find an interval I on which Rolle's Theorem applies to $f(x) = x^3 - 7x^2 + 10x$. Then find all points c in I at which $f'(c) = 0$. Confirm with your graphing calculator.

Practice:

1. Determine whether Rolle's Theorem applies to the following functions on the given interval. If so, find the point(s) guaranteed to exist by Rolle's Theorem.

a. $f(x) = x^2 - 4x + 1, \quad [0, 4]$

b. $f(x) = x^3 - 3x^2 + 2x + 5, \qquad [0,\ 2]$

c. $f(x) = x\sqrt{x+6}, \qquad [-6,\ 0]$

2. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1,\ 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

Determine whether Rolle's Theorem applies to the following functions on the given interval. If so, find the point(s) guaranteed to exist by Rolle's Theorem.

1. $f(x) = x(x-1)^2; [0,1]$

2. $f(x) = \sin 2x; [0, \pi/2]$

3. $f(x) = \cos 4x; [\pi/8, 3\pi/8]$

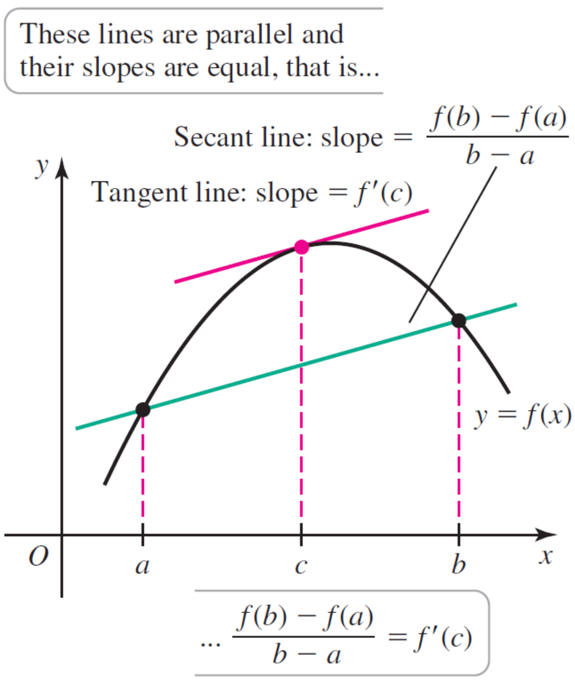
4. $f(x) = 1 - |x|; [-1,1]$

5. $f(x) = x^3 - 2x^2 - 8x; [-2,4]$

4.2 Mean Value Theorem

Mean Value Theorem

The Mean Value Theorem is easily understood with the aid of a picture. The figure below shows a function f differentiable on (a, b) with a secant line passing through $(a, f(a))$ and $(b, f(b))$; the slope of the secant line is the average rate of change of f over $[a, b]$. The Mean Value Theorem claims that there exists a point c in (a, b) at which the slope of the tangent line at c is equal to the slope of the secant line. ***In other words, we can find a point on the graph of f where the tangent line is parallel to the secant line.***



THEOREM 4.4 Mean Value Theorem
If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The following situation offers an interpretation of the Mean Value Theorem. Imagine driving for 2 hours to a town 100 miles away. While your average speed is $100\text{mi}/2\text{hr} = 50\text{mi}/\text{hr}$, your instantaneous speed (measured by the speedometer) almost certainly varies. The Mean Value Theorem says that at some point during the trip, your instantaneous speed equals your average speed, which is $50\text{mi}/\text{hr}$.

EXAMPLE: Determine whether the function $f(x) = 2x^3 - 3x + 1$ satisfies the conditions of the Mean Value Theorem on the inter-val$[-2, 2]$. If so, find the point(s) guaranteed to exist by the theorem.

Consider the following functions on the given interval $[a, b]$.

a. Determine whether the Mean Value Theorem applies to the following functions on the given interval $[a, b]$.
b. If so, find the point(s) that are guaranteed to exist by the Mean Value Theorem.

1. $f(x) = 7 - x^2; [-1,2]$

2. $f(x) = x^3 - 2x^2; [0,1]$

3. $f(x) = \frac{1}{(x-1)^2}; [0,2]$

4. $f(x) = e^x; [0,1]$

5. $f(x) = \ln 2x; [1, e]$

6. $f(x) = \sin x; \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

7. $f(x) = 3x^2 + 2x + 5, \quad [-1,\ 1]$

8. $f(x) = e^{-2x}, \quad [0,\ 3]$

9. $f(x) = \dfrac{x}{x+2}, \quad [1,\ 4]$