Calculus H Name ________________

Derivatives Ch. 3

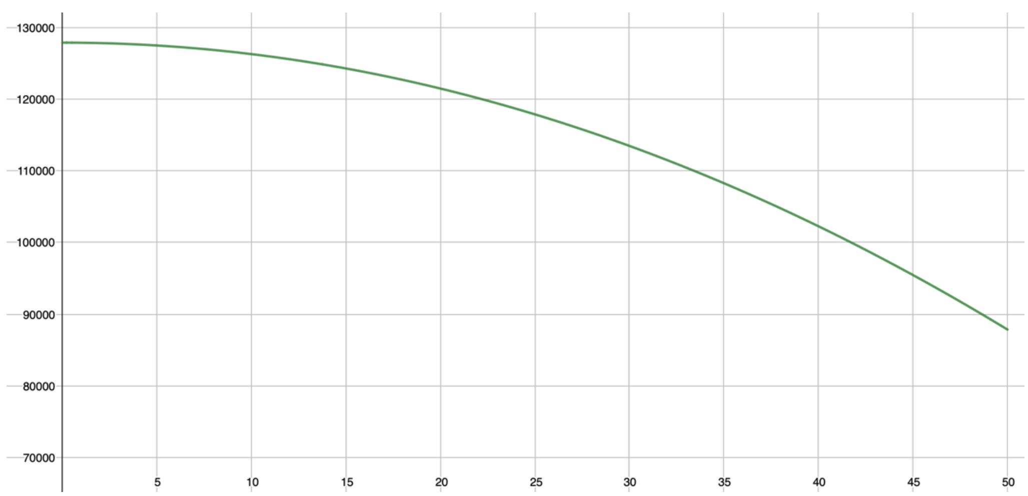
Can a Human Break the Sound Barrier?

On October 14th, 2012, Austrian skydiver Felix Baumgartner broke a world record for a high-altitude dive when he ascended 127,850 feet in a helium balloon and then went into a free fall lasting more than 4 minutes.

1. Baumgartner is in free fall for 4 minutes and 20 seconds (260 seconds) before he deploys his parachute at an elevation of 8,420 feet above sea level.
   a. What was the vertical distance of the freefall?

   b. What was his average velocity during the freefall?

2. His elevation (in feet) above sea-level, t seconds after stepping off the balloon can be approximated by $f(t) = 127850 - 16t^2$ for $0 \leq t \leq 50$.
   a. Look at the graph of $f(t)$ below. Label both axes.



   b. Was Baumgartner traveling at a constant velocity? How do you know?

   c. What time does it look like Baumgartner is traveling the fastest? How can you tell?

3. Let's see if we can estimate his velocity exactly 30 seconds after leaving the balloon.
    a. What is his average velocity between $t = 20$ and $t = 30$? Show your work.

       Is this faster or slower than the velocity at exactly 30 seconds? Explain.

    b. What is his average velocity between $t = 30$ and $t = 40$? Show your work.

       Is this faster or slower than the velocity at exactly 30 seconds? Explain.

4. Let's take an interval even closer to 30.
    a. Find the average velocity between $t = 29$ and $t = 30$. Show your work.

    b. Find the average velocity between $t = 30$ and $t = 31$. Show your work.

5. Are the estimates in 4a and 4b better or worse than those in 3a and 3b? Why?

6. How could we get an even better estimate?

7. We're going to find the average velocity between $t = 30$ and $t = 30 + h$. Let's break it down into steps.

   a. Find $f(30 + h)$. Simplify.

   b. Find $f(30 + h) - f(30)$.

   c. Write the expression for $\frac{f(30+h)-f(30)}{h}$ using what you found above.

   d. What value of h would represent his velocity at *exactly* $t = 30$? Explain.

   e. Show how you could determine this velocity.

8. The speed of sound is 1,125.3 feet per second. Did Baumgartner go supersonic?

3.1 The Derivative of a Function at a Point

The **derivative of *f* at *a*,** denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and *a* is in the domain of *f*.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $f'(a)$ exists, we say the *f* is **differentiable** at *a*.

1. Given the function $f(x) = \dfrac{3}{x}$,

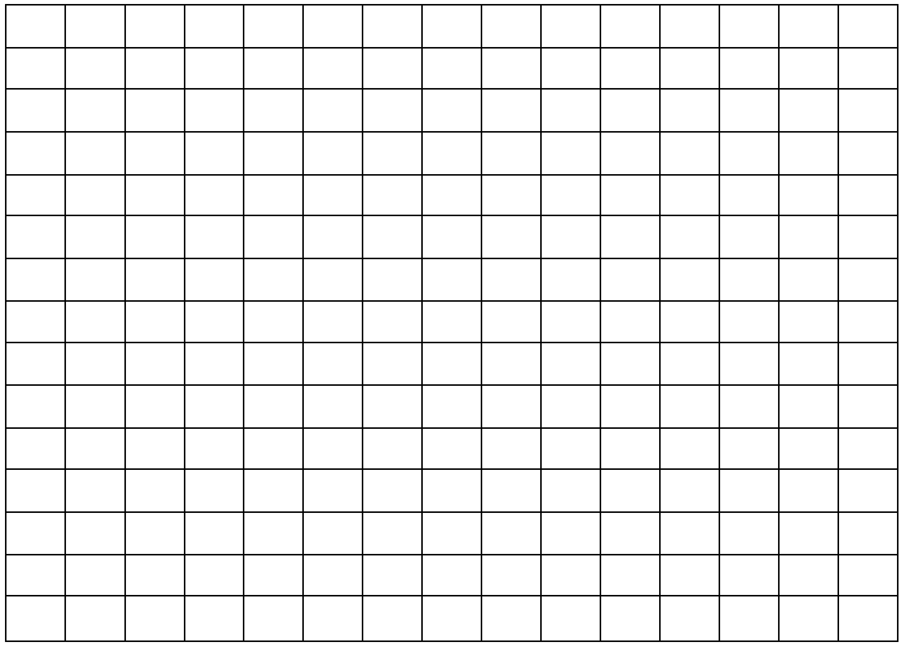
a. Use the limit definition at a point to find $f'(2)$.

b. What does the derivative at x = 1 represent for the graph of f(x)?

c. Write the equation of the tangent line at $\left(2, f(2)\right)$ with slope $f'(2)$.

2. Consider the function $f(x) = 4x - x^2$.

a. Draw the function on the interval $[0, 6]$.



b. Draw tangent lines where x = 1 and 2.

c. Use the tangent lines to estimate the instantaneous rate of change at each x value.

$f'(1) \approx$ ______ $f'(2) \approx$ ______

d. Use the limit definition at a point to find $f'(1)$ and $f'(2)$.

e. Find the equation of the tangent line at x = 1 and x = 2.

3.2 The Derivative Function

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say the f is **differentiable** at x. If f is differentiable at every point of the open interval I, we say that f is differentiable on I.

Find the derivative $f'(x)$ for $f(x) = 3x^2 - 2x$

Using the above, find $f'(0)$, $f'\left(\frac{1}{3}\right)$ and $f'(1)$. What do each of these tell you about the curve f(x)?

Derivative Notation

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}$$

Other common ways of writing the derivative

$$\frac{df}{dx} \qquad \frac{d}{dx}\big(f(x)\big) \qquad D_x\big(f(x)\big) \qquad y'(x)$$

The following notation represents the derivative of f evaluated at a.

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Let y = f(x) = $\sqrt{x}$

Compute $\frac{dy}{dx}$

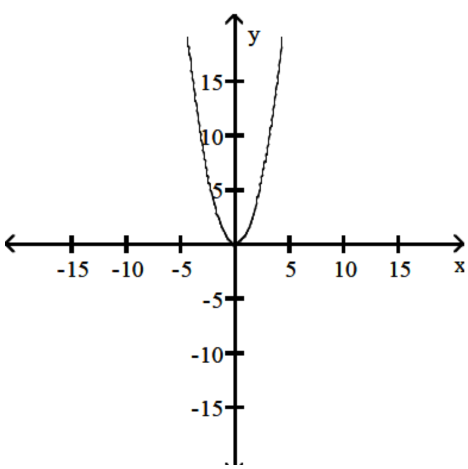
Find the equation of the tangent line when x = 4.

Find $g'(t)$ when $g(t) = \dfrac{1}{t^2}$

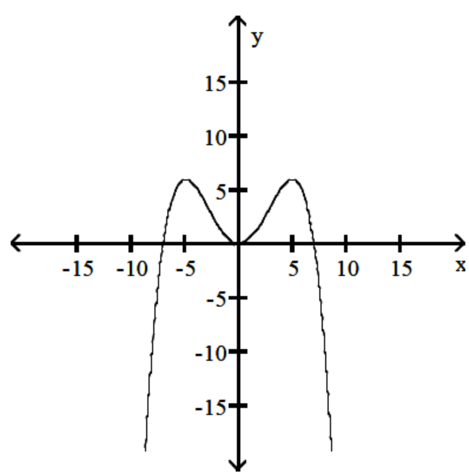
Graphing Derivatives

1. Graph the derivative of the following function.

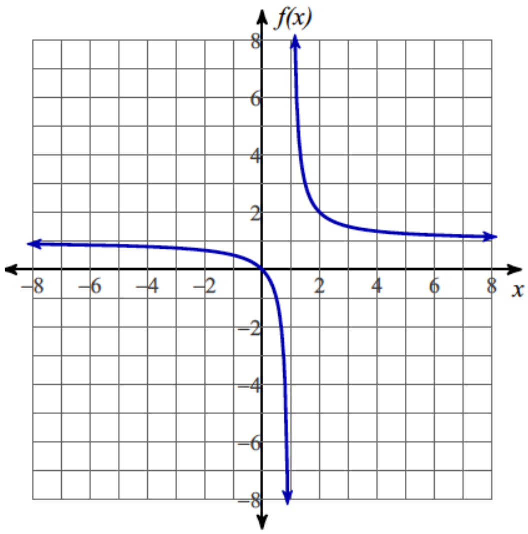
a.



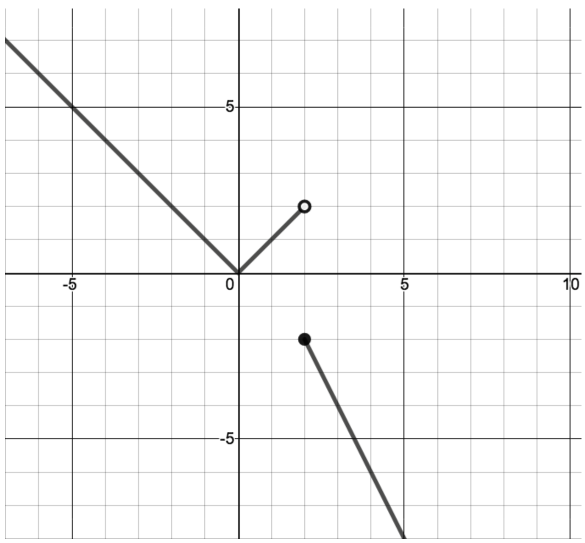
b.



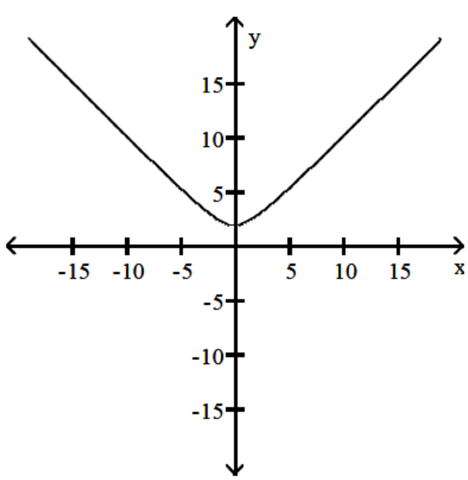
c.



d.



e.



Differentiable Implies Continuous
If f is differentiable at x = a, then f is continuous at x = a. Differentiability implies Continuity.

Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.

But continuity does not imply differentiability.

Let's look at $f(x) = |x|$

Local Linearity and Differentiability

If a function is differentiable at a point then the function is locally linear at that point. This means that if you zoom way, way in on a function at a point where it is differentiable eventually you get a graph that is all but indistinguishable from a a line – in fact it's the tangent line at that point that the function is indistinguishable.

Lets look at:

$f(x) = |x|$

$f(x) = \sin(x)$

Lets look at $f(x) = x^{\frac{1}{3}}$

When is a function not differentiable at a point?

1. *f* is not continuous at *a.*
2. *f* has a corner/sharp turn at *a.*
3. *f* has a vertical tangent at *a.*

Derivatives 3.3

The Constant Rule

If c is a real number, then $\frac{d}{dx}(c) = 0$

The Power Rule

If n is any real number, then $\frac{d}{dx}(cx^n) = nx^{n-1}$

The Constant Multiple Rule

If f is differentiable at x and c is a constant, then $\frac{d}{dx}(cf(x)) = cf'(x)$

The Sum and Difference Rule

If f and g are differentiable at x, then $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

1. Find the derivative function $f'(x)$ if $f(x) = 6x^{11} - 4x^9 + 2x^4 - 8x + 23$

2. Find the derivative function $g'(x)$ if $g(x) = x^{12} - \frac{3}{2}x^{10} + 6x^{\frac{3}{2}} - \sqrt{7}x + e^2$

3. Find the derivative function $h'(x)$ if $h(x) = \sqrt[4]{x^3} + \sqrt{x}$

4. Find the derivative function $f'(x)$ if $f(x) = \frac{1}{x^5} + \frac{1}{\sqrt[3]{x}}$

The number e

The number e = 2.718281828459 … satisfies

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

It is the base of the natural exponential function f(x) = e^x.

The Derivative of e^x

The function f(x) = e^X is differentiable for all real numbers x, and

$$\frac{d}{dx}(e^x) = e^x$$

5. Write an equation of the line tangent to the graph of $f(x) = 2x - \frac{e^x}{2}$ at the point (0, $-\frac{1}{2}$)

6. Find the point(s) on the graph of f(x) (from #5) at which the tangent line is horizontal.

7. Let f(x) = $2x^3 - 15x^2 + 24x$. For what values of x does the line tangent to the graph of f have a slope of 6?

Higher – Order Derivatives

Assuming y = f(x) can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}\left(f'(x)\right)$$

For integers $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

Other common notations for the second derivative of y = f(x) include $\frac{d^2y}{dx^2}$ and $\frac{d^2f}{dx^2}$.

The notations $\frac{d^ny}{dx^n}$, $\frac{d^nf}{dx^n}$, and $y^{(n)}$ are used for the nth derivatives of f.

8. Find the third derivative of the following functions

a. $f(x) = 3x^3 - 5x + 12$

b. $y = 3t + 2e^t$

Derivatives 3.4

Let's Explore how to differentiate the product of two functions!

1. Let $g(x) = x^7$ and let $h(x) = x^{11}$.

Find $g'(x)$.

Find $h'(x)$.

2. Let $f(x) = g(x) \bullet h(x)$ .Write an equation for $f(x)$ as a single power of x. f(x) = ______

Find $f'(x)$.

3. True or False? $f'(x) = g'(x) \bullet h'(x)$ Show work to support your answer.

Product Rule

If *f* and *g* are differentiable at *x*, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + g'(x)f(x)$$

Find and simplify the following derivatives.

1. $\frac{d}{dv}\left(v^2\left(2\sqrt{v}+1\right)\right)$

2. $\frac{d}{dx}\left(x^2e^x\right)$

Quotient Rule

If *f* and *g* are differentiable at *x*, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-g'(x)f(x)}{\left(g(x)\right)^2}$$

Find and simplify the following derivatives.

3. $\dfrac{d}{dx}\left(\dfrac{x^2+3x+4}{x^2-1}\right)$

4. $\dfrac{d}{dx}\left(e^{-x}\right)$

5. Find the equation of the line tangent to the graph of $f(x)=\dfrac{x^2+1}{x^2-4}$ at the point (3, 2).

Find the following derivatives.

6. $\frac{d}{dz}\left(\sqrt[3]{z}e^{z}\right)$

7. $\frac{d}{dx}\left(\frac{3x^{\frac{5}{2}}}{2x^{2}+4}\right)$

8. $y=\frac{4xe^{x}}{x^{2}+1}$

Derivatives 3.5

Using your calculators, find the following limits. (Make sure you are in radians)

$$\lim_{x\to 0}\frac{\sin x}{x} = \_\_\_\_\_\_ \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = \_\_\_\_\_\_$$

Evaluate the following limits using the above.

1. $\lim_{x\to 0}\frac{\sin 4x}{x} =$

2. $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} =$

3. $\lim_{x\to 0}\frac{\tan 2x}{x} =$

Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Calculate $\frac{dy}{dx}$ for the following functions.

4. $y = e^x \cos x$

5. $y = \sin x - x\cos x$

6. $y=\dfrac{1+\sin x}{1-\sin x}$

Derivatives of the Trigonometric Functions

$\dfrac{d}{dx}(\sin x)=\cos x$

$\dfrac{d}{dx}(\cos x)=-\sin x$

$\dfrac{d}{dx}(\cot x)=-\csc^2 x$

$\dfrac{d}{dx}(\tan x)=\sec^2 x$

$\dfrac{d}{dx}(\csc x)=-\csc x\cot x$

$\dfrac{d}{dx}(\sec x)=\sec x\tan x$

7. Find the derivative of $y=\sec x\csc x$

8. Find $\frac{d^2y}{dx^2}$ of y = csc x

9. Find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$ of y = cos x

10. Find $\frac{d^{40}y}{dx^{40}}$ and $\frac{d^{42}y}{dx^{42}}$ of y = sin x

Derivatives 3.6 Velocity

Displacement is the difference in the object's position from one time to another.

Position is the location of the object at a specified time.

Velocity tells us speed and direction. It is the instantaneous rate of change of the position function with respect to time. Or the derivative of the position (displacement) function.

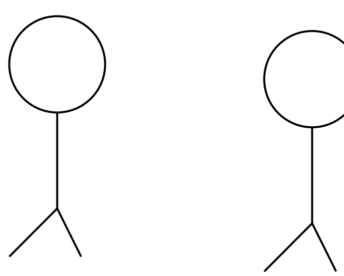
Speed is the absolute value of velocity. It tells us how fast an object is going but not the direction.

Acceleration is the instantaneous rate of change of the velocity with respect to time. It also has a magnitude and direction.

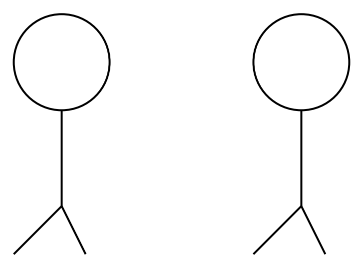
- Units are usually $ft/\sec^2$. The unit of time is squared.
- Acceleration due to gravity is $9.8\,m/\sec^2$ *or* $32\ ft/\sec^2$.
- Acceleration is the first derivative of velocity or the second derivative of position (displacement).

Relationship between speeding up, velocity and acceleration:

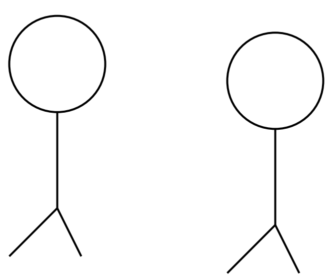
$v(t) > 0$ and $a(t) > 0$, speeding up



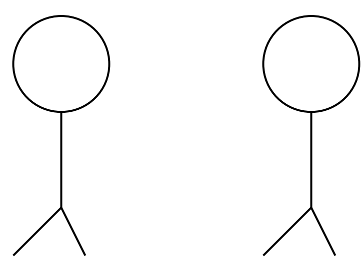
$v(t) > 0$ and $a(t) < 0$, slowing down



$v(t) < 0$ and $a(t) > 0$, slowing down



$v(t) < 0$ and $a(t) < 0$, speeding up



If velocity and acceleration have the same sign (are in the same direction) then the object will be speeding up.

Example 1: A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.

a. Find the displacement of the particle during the first 2 seconds.

b. Find the average velocity of the particle during the first 4 seconds.

c. Find the instantaneous velocity of the particle when t = 4.

d. Find the acceleration of the particle when t = 4.

e. Describe the motion of the particle. At what values of t does the particle change directions?

Example 2. Suppose we know that $a(t) = 4t - 5$, we then know that $v'(t) = 4t - 5$.

a. What is $v(t)$?

b. What is the relationship between $a(t)$ and $v(t)$?

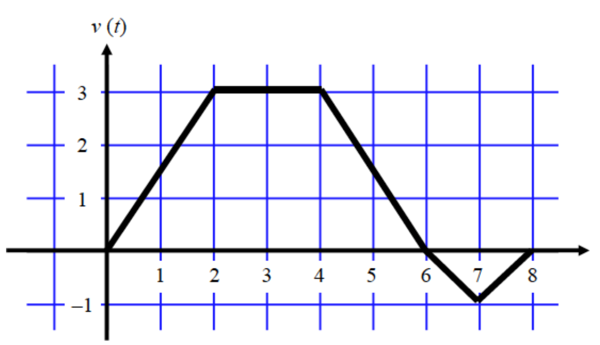
Example 3. Once again trying to blow up earth because it interferes with his view of Venus, Marvin the Martian lands on the moon. Bugs Bunny, as always, interferes with his plan. Chasing Bugs, Marvin fires a warning shot straight up into the air with his Acme Disintegration Pistol. The height (in feet) after t seconds of the shot is given by $s(t) = -2.66t^2 + 135t + 3$.

a) Find the velocity and acceleration as functions of time.

b) What is the position of the shot when the velocity is 0?

Example 4. A bug begins to crawl up a vertical wire at time t = 0. The velocity, v, of the bug at time t, $0 \le t \le 8$ is given by the function whose graph is shown below.

a) At what value of t does the bug change direction?



b) During which time intervals in the bug slowing down?

Practice 1

1. What is the relationship between position, velocity, and acceleration?

2. Fill in the blanks.

a) When the ______________________ is positive, the object is moving in a positive direction.

b) An object is __________________________ when the velocity and acceleration have different signs.

c) An object is stopped when ____________________ is zero.

d) Speed is always positive because it is the _______________________________ of velocity.

A particle moves along a horizontal line. Its position function is s(t) for $t \geq 0$. Find the following:

3. If $s(t) = -t^4 + 15t^3$, find the velocity function v(t) and the acceleration function a(t).

4. If $s(t) = t^4 - 8t^3$, find the times t when the particle changes direction.

5. If $s(t) = t^2 - 4t - 96$, find the times t when the acceleration is 0.

6. If $s(t) = -t^2 + t + 72$, find the intervals of time when the particle is slowing down and speeding up.

7. If $s(t) = -t^3 + 10t^2$, find the position, velocity, speed, and acceleration at t = 7.

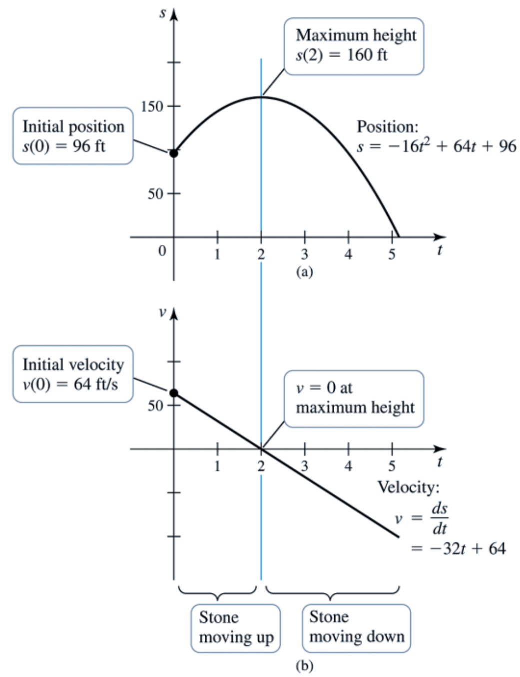
8. If $s(t) = -t^3 + 10t^2$ what is the distance traveled from t = 0 to t = 8?

9. If $s(t) = t^3 - 23t^2 + 120t$, find the intervals of time when the particle is slowing down and speeding up.

10. If $s(t) = -t^4 + 11t^3$, find the position, velocity, speed and acceleration at t = 4.

Example 5. Motion in a gravitational field. Suppose a stone is thrown vertically upward with an initial velocity of 64ft/s from a bridge 96 ft above a river. By Newton's laws of motion, the position of the stone (measured as the height above the river) after t seconds is $s(t) = -16t^2 + 64t + 96$ where $s = 0$ is the level of the river.

a. Find the velocity and acceleration functions.



b. What is the highest point above the river reached by the stone?

c. With what velocity will the stone strike the river?

Practice 2

1. A particle moves along the x–axis so that its position at any time $t \geq 0$ is given by the function $x(t) = t^3 - 12t + 1$, where x is measured in feet and t is measured in seconds. Justify each response and indicate units of measure when appropriate.

a) Find the displacement during the first 3 seconds.

b) Find the average velocity during the first 3 seconds.

c) Find the distance the object traveled $0 \leq t \leq 3$

d) Find the instantaneous velocity at t = 3 seconds.

e) Find the acceleration when t = 3 seconds.

f) When is the particle moving left?

g) At what value(s) t does the particle change direction?

h) When is the particle speeding up? slowing down?

2. Let $s(t) = t^3 - 6t^2$ for $t \geq 0$.

a. Make a table showing the position, velocity, speed, and acceleration of the particle at times t = 0, t=1, t=2, t=3, t=4, and t = 5.

b. At each of these times, specify the direction of motion (forward/backward, up/down), if any, and whether the particle is speeding up, slowing down, or neither.

3. Let $s(t) = t^3 - 9t^2 + 24t$ for $t \geq 0$.

a. Find all times in which the particle is at rest (velocity = 0)

b. At what values of t is the particle moving backward?

c. At what values of t is the particle moving forward?

d. Find all times in which the particle's speed is constant (not accelerating).

Example 6. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

a. Find the rock's velocity and acceleration as a function of time. (the acceleration in this case is the acceleration on the moon)

b. How long did it take the rock to reach its highest point?

c. How high did the rock go?

d. How long did it take the rock to reach half its maximum height?

e. How long was the rock aloft?

f. Find the rock's speed when hitting the moon.

Example 7. A ball is dropped from the top of the Washington Monument which is 555 feet high.

a. How long will it take for the ball to hit the ground? (Use the fact that acceleration due to gravity on the earth is approximately 32 ft/sec^2)

b. Find the ball's speed at impact.

Practice 3.

For problems 1 and 2, find equations for the velocity, v(t) and acceleration a(t) of a moving object if y(t) is its displacement.

1. $y(t) = 5t^4 - 3t^{2.4} + 7t$

2. $y(t) = 0.3t^{-4} - 5t$

3. An object moves with displacement $x(t) = -t^3 + 13t^2 - 35t + 27$, where x is in feet and t is in seconds.

a. Find equations for the velocity, v(t), and the acceleration, a(t).

b. Find the velocity and acceleration at t = 1, t = 6, and t = 8. At each time, state the object is speeding up or slowing down.

4. An object moves with displacement $x(t)=t^4-11t^3+38t^2-48t+50$, where x is in feet and t is in seconds.

a. Find equations for the velocity, v(t), and the acceleration, a(t).

b. Find the velocity and acceleration at t = 1, t = 3, and t = 5. At each time, state the object is speeding up or slowing down.

5. Using $g(x)=3x-1$, $h(x)=3x-5$, and $f(x)=x^2+4x$ find the following.

a. g(h(x))

b. f(h(x))

c. g(f(x))

6. In each of the following, write formulas for f(x) and g(x) so that h(x) = f(g(x)).

a. $h(x)=\sin 3x$

b. $h(x)=\sin^3 x$

c. $h(x)=\sin x^3$

d. $h(x)=2^{\cos x}$

e. $h(x)=\dfrac{1}{\tan x}$

f. $h(x)=\log(\sec x)$

Derivatives 3.7

If f and g are functions, then their **composite** $f \circ g$ is the function with $[f \circ g](x) = f[g(x)]$ for each x in the domain of g such that $g(x)$ is in the domain of f.

Example 1: If $f(x) = 2x+1$ and $g(x) = \frac{1}{x}$, find $[f \circ g](x)$ and $g\left(f(x)\right)$. Describe the domain.

Example 2: Given the following composite function $h(x) = \sqrt{\sin(2x)}$, determine the 3 functions $g(x), f(x), k(x)$ such that $h(x) = [k \circ f \circ g](x) = k\left(f\left(g(x)\right)\right)$.

Example 3: Given tables for functions f and g.

x	*f(x)*	*x*	*g(x)*
-1	2	-1	3
0	4	0	4
1	3	1	2
2	0	2	6
3	1	3	2
4	-1	4	-1

Find:

$f\left(g(3)\right)$

$[g \circ f](2)$

$f\left(f(4)\right)$

$[g \circ g](4)$

all inputs x such that $f\left(g(x)\right) = 2$

Example 4: Let $f(x)=2x-3,\ g(x)=e^{x},$ and $h(x)=\ln x$ Find a formula for each function.

$f\left(f(x)\right)$

$[f\circ g](x)$

$h\left(g(x)\right)$

$g\left(h(x)\right)$

Example 5. In each of the following, write formulas for $f(x)$ and $g(x)$ so that $h(x)=f(g(x))$.

$h(x)=(x+4)^{3}$

$h(x)=e^{x-1}$

$h(x)=\ln(2x+5)$

$h(x)=\dfrac{1}{(2x-1)^{2}}$

$h(x)=\sqrt{x+3}-\sqrt[3]{x+3}$

The Chain Rule

If y is a differentiable function of u and u is a differentiable function of x, then the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Method 1

1. Identify an outer function f and an inner function g, and let $u = g(x)$.
2. Replace $g(x)$ with u to express y in terms of u:

   $y = f(g(x)) = f(u)$
3. Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$.
4. Replace with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

OR

Method 2

If $h(x) = f(g(x))$ then $h'(x) = f'(g(x)) \times g'(x)$

OR

Method 3

Outside function, inside function

a. Take the derivative of the outer function and keep the inner function (plug in the inner function).
b. Take the derivative of the inner function
c. Multiply the result of step 1 and step 2.

Examples: Use the chain rule to differentiate.

1. y = cos (4x)

2. $h(x) = \sec(x^4)$

3. $y = \sin^2(x)$

4. f(x) = csc(sin(x))

5. $y = \tan(\sqrt{x} + 2)$

6. $h(x) = (5x - 3)^2$

7. $k(x)=\left(x^2-5\right)^3$

8. $q(x)=\left(5x^2-3x+4\right)^{100}$

9. $g(x)=\sqrt{5x^2+1}$

10. h(x) = $\left(\frac{5t^2}{3t^2+2}\right)^3$

11. $y = e^{-3x}$

12. Calculating derivatives at a point. Let $h(x) = f(g(x))$. Use the values in the table below to calculate $h'(1)$ and $h'(2)$.

x	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	3
2	7	1	4

13. Find $\frac{d}{dx}(\tan x + 10)^{21}$.

14. Find $\frac{d}{dx}\sin(e^{\cos x})$.

15. Find $\frac{d}{dx}\left(x^2\sqrt{x^2+1}\right)$.

Derivatives 3.8 Implicit Differentiation

Explicit functions: can be solved for *y* without "resorting to cases." This is because y is defined explicitly in terms of x.

Ex. $y = 3x + 5$

Implicit functions either cannot be solved for *y* or cannot be solved for *y* without resorting to some cases. This is because the relations are implied by an equation.

Ex. $x^3y^2 - 5xy\left(3x + 7y^5\right) = 8$

Think about how you would graph a circle on your calculator.

Implicit Form	Explicit Form	Derivative
$x^2y = 2$	$y = \dfrac{2}{x^2}$	

Sometimes working with implicit functions is so much easier that you wouldn't even bother trying to solve for *y*.

Implicit Differentiation

To find $\frac{dy}{dx}$ *for a relation whose equation is written implicitly:*

1. Differentiate both sides of the equation with respect to x. Obey the chain rule by multiplying by $\frac{dy}{dx}$ each time you differentiate an expression containing y.

2. Isolate $\frac{dy}{dx}$ by getting all of the $\frac{dy}{dx}$ terms onto one side of the equation, and all other terms onto the other side. Then factor, if necessary, and divide both sides by the coefficient of $\frac{dy}{dx}$.

Derivatives of implicit functions really just use the chain rule over and over and over…then you solve for $\frac{dy}{dx}$.

Constantly say this sentence to yourself as you take the derivative: "but y is a function of x so I have to chain rule this thing…"

Example 1:
Find the derivative of $x^2 + y^2 = 1$

Example 2:
Find the derivative of $y^3 + x^2y^5 - 8x^5 = 24$

Example 3:
Find the derivative of $sin(x \cdot y) = x^2 + y$

Example 4:
Find the derivative of $x^3 y = 5$

Example 5:

Find the equations of the tangent lines to the curve $x^3 + y^2 = 5$ at $x = -3$

Logarithms

Where does a logarithm come from?

So try some of these without a calculator.

$\log_3 27$ $\log_2 32$ $\log_{10} \frac{1}{100}$ $\log_3 1$ $\log_2 2$ $\ln 1$

Common Log & Natural Log

1. Change of Base Property

Examples:

$\log_6 9$ to an expression with common logarithms

$\log_{11} 4$ to an expression with natural logarithms

$\log_{25} 125$ to an expression with logarithms with base 5

2. Property for $\log(ab)$

Expand:

$a.$ $\log_5 6x$

$b.$ $\log_7 12xyz$

Condense into a single expression:

$a.$ $\log_5 6 + \log_5 2$

$b.$ $\log_2 4 + \log_2 5 + \log_2 x$

$c.$ $\log_6 8 + \log_6 3 + \log_6 x + \log_6 y$

3. Property for $\log\left(\frac{a}{b}\right)$

Expand:

$a.$ $\log_9 \frac{4}{7}$

$b.$ $\log_{10} \frac{x}{y}$

$c.$ $\log_2 \frac{3}{z}$

Condense into a single expression:

$a.$ $\log_2 10 - \log_2 5$

$b.$ $\log_4 6 - \log_4 12$

$c.$ $\log_6 x - \log_6 y$

$d.$ $\log_5 10 - \log_5 2$

Let's try a combination of both properties

Expand (Hint: You may need to use parenthesis)

$a.\ \log_{10} 2 + \log_{10} 3 - \log_{10} 4 + \log_{10} 3$

$b.\ (\log_{10} 2 + \log_{10} 3) - (\log_{10} 4 + \log_{10} 3)$

$c.\ \log_{10} x + \log_{10} y - \log_{10} z - \log_{10} a$

$d.\ \log_{10} 2 + \log_{10} x - (\log_{10} 4 + \log_{10} 7)$

$e.\ \log_{10} 3 - \log_{10} 4 + \log_{10} 5$

Condense: Work from left to right!

$$a.\ \log_{10} \frac{6x}{7}$$

$$b.\ \log_{4} \frac{2x}{3y}$$

$$c.\ \log_{8} \frac{50}{xy}$$

$$d.\ \log_{10} \frac{2xy}{7ab}$$

4. Property for $\log\left(a^b\right)$

Practice!

Expand:

a. $\log_2 x^4$

b. $\log_4 \sqrt{y}$

c. $\log_6 \sqrt[3]{z}$

d. $\log_5 x^{10}$

Condense into a single expression:

a. $2\log_{10} x$

b. $3\log_{10} x$

c. $\frac{1}{2}\log_2 81$

d. $\frac{1}{5}\log_{10} 32$

Derivative of Exponential and Logarithmic Functions

If $f(x) = b^x$, then $f'(x) = b^x \ln b$

Find the derivative for each of the following.

1. $y = p^x$

2. $f(x) = 3^x$

3. $g(t) = 108 * 2^{t/12}$

Let's find the derivative of $y = \ln x$

Find the derivatives.

1. $y = \ln\left(x^4\right)$

2. $y = \ln\left(\sin\left(x\right)\right)$

3. $y = \ln\left(\cos\left(x^2\right)\right)$

4. Find $\frac{d}{dx}\left(\ln\frac{x^6}{\sin^2 x}\right)$

5. Find $\frac{d}{dx}\left(\ln\frac{\cos^2 x}{x^3}\right)$

6. Find $\frac{d}{dx}\left(\ln\left(x^2 \cos x\right)\right)$

7. Find $\frac{d}{dx}\left(\frac{\log_4 x}{\log_2 x}\right)$

Let's find the derivative of $y = log_b x$

Find the derivatives.

1. $f(x) = log_5(2x + 1)$

2. $T(n) = nlog_2 n$

Logarithmic Differentiation

Differentiate $y = 8^x$

So why would anyone ever do this…this technique will be useful when differentiating functions where the base itself is a variable.

Example 1:

Find $f'(x): f(x) = \left(x^3 + 4\right)^{\cos x}$

Example 2:

If $f(x) = \dfrac{(3x+7)^5}{\sqrt[3]{x+2}}$, *find* $f'(x)$

Differentiate the functions below.

1. $y = \dfrac{x^5}{(1-10x)\sqrt{x^2+2}}$

2. $y = x^x$

3. $y=\frac{\left(x^2-9\right)^{10}\left(2x+4\right)^6}{\sin x}$

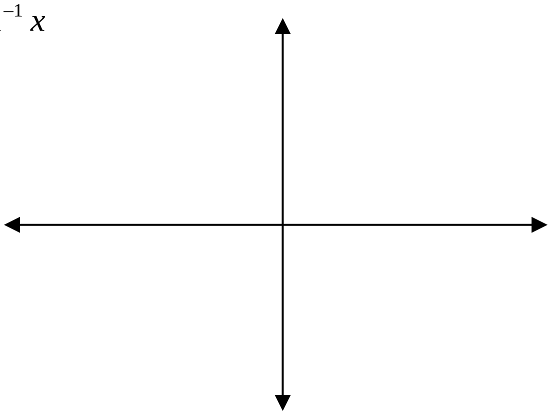
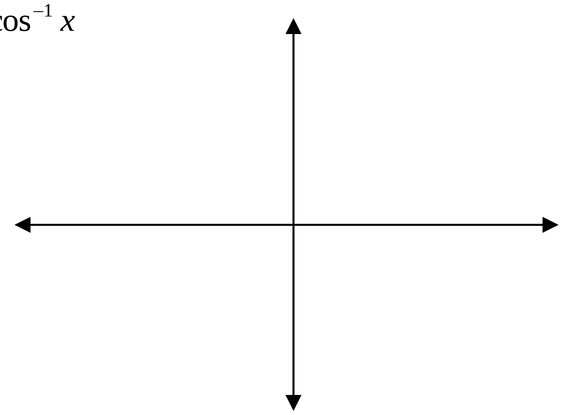
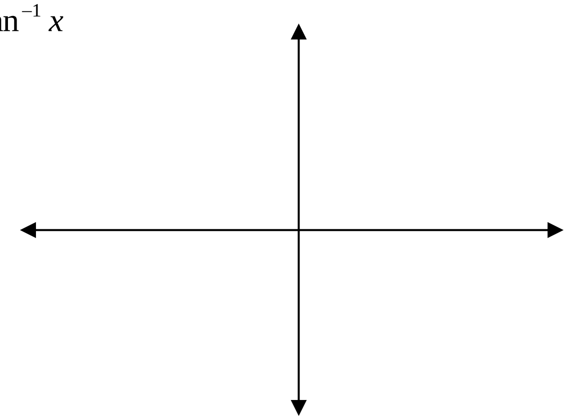
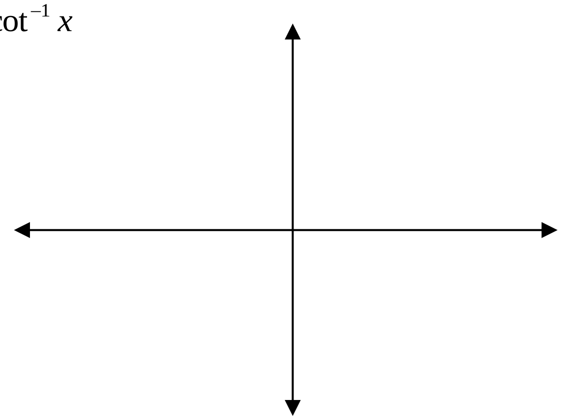
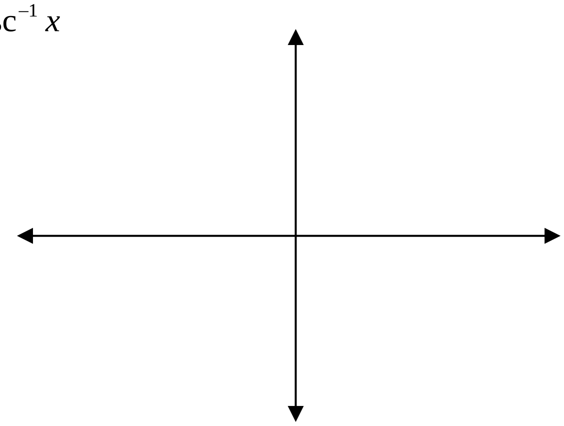
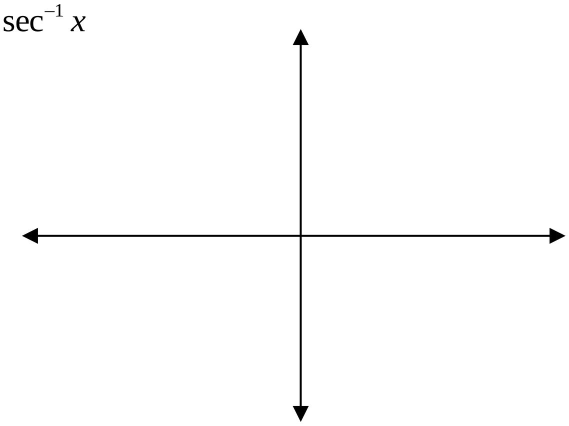
4. $y=\left(1-3x\right)^{\cos x}$

5. $y=x^{\sqrt{x}}e^{3x+4}$

Derivatives 3.10 Derivatives of Inverse Trig. Functions

Inverse Trig Functions Review

Draw a sketch of each inverse function and identify the range.

$y = \sin^{-1} x$ 	$y = \cos^{-1} x$ 
$y = \tan^{-1} x$ 	$y = \cot^{-1} x$ 
$y = \csc^{-1} x$ 	$y = \sec^{-1} x$ 

Notes about Inverses:

- ✓
- ✓
- ✓
- ✓

Examples:

1. $y = \sin^{-1}(6x)$

2. $f(x) = \cot^{-1}\left(\dfrac{x^2}{2}\right)$

3. Evaluate $f'(2\sqrt{3})$, where f(x) = $f(x) = x\tan^{-1}\left(\frac{x}{2}\right)$

4. Find the equation of the line tangent to the graph of $g(x) = \sec^{-1} 2x$ at the point $\left(1, \frac{\pi}{3}\right)$

5. $y = \tan^{-1}\left(\frac{x}{7}\right)$

6. $y = \sec^{-1}\left(2x^3\right)$

7. $y = \sin^{-1}\left(e^{3x}\right)$

8. $y = \cos^{-1}\left(\ln x\right)$

9. $y = \csc^{-1}\left(\frac{x}{6}\right)$

10. $y = \cot^{-1}\left(x^5\right)$

Derivatives 3.11 Related Rates

Let's now look at derivatives as rates of change in problems in which the variables change with respect to time. The essential feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time. Here are two examples illustrating this type of problem.

1. An oil rig springs a leak in calm seas, and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 meters?

Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

2. Two small planes approach an airport, one flying due west at 120mi/hr and the other flying due north at 150mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 225 miles from the airport?

3. Morning coffee. Coffee is draining out of a conical filter at a rate of $2.25\text{in}^3/\text{min}$. If the cone is 5 in tall and has a radius of 2 in , how fast is the coffee level dropping when the coffee is 3 in deep?

Practice:

1. A ladder 20 feet long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 12 feet above the ground?

2. The radius of a sphere is increasing at a constant rate of 0.5 inch/second.
   a. When the radius of the sphere is 15 inches, at what rate is the volume of the sphere changing?
   b. When the volume and radius of the sphere are changing at the same rate, what is the radius of the sphere?

4. A balloon is being inflated at a rate of $10\pi \frac{\text{ft}^3}{\text{sec}}$. At what rate is the radius increasing when $r = 2$ feet?

5. Sheila walks to Lake Menomin and throws a rock into the lake. Since the lake is calm, ripples in the shape of concentric circles are formed on the water. If the radius of the outer ripple is increasing at a rate of 2 feet per second, at what rate is the total area of disturbed water changing when the radius is 5 feet?

6. A 6-meter ladder is against a wall. If its bottom is pulled at a constant rate of $\frac{1}{2} m/\text{sec}$, how fast is the ladder top sliding when it reaches 5 meters up the wall?

7. A winch (altitude of 20 feet) reels in a rope at a rate of 2 ft/ sec. How fast is the boat moving when the rope is 45 feet?

8. The edges of a cube are increasing at a rate of 2 cm/s.
   a. How fast is the volume of the cube increasing when each edge is 5 cm long?

   b. How fast is the surface area of the cube changing when each edge is 5 cm?

9. Water leaking onto a floor forms a circular pool. The radius of the pool increases at a rate of 4 cm/min. How fast is the area of the pool increasing when the radius is 5 cm ?

10. Oil spilling from a ruptured tanker spreads in a circle on the surface of the ocean. The area of the spill increases at a rate of $9\pi\ \text{m}^2/\text{min}$. How fast is the radius of the spill increasing when the radius is 10 m ?

11. A conical paper cup is 10 cm tall with a radius of 10 cm . The cup is being filled with water so that the water level rises at a rate of 2 cm/sec. At what rate is water being poured into the cup when the water level is 8 cm ?

12. A spherical balloon is inflated so that its radius (r) increases at a rate of $\frac{2}{r}$ $\mathrm{cm/sec}$. How fast is the volume of the balloon increasing when the radius is 4 cm ?

13. A 7 ft tall person is walking away from a 20 ft tall lamppost at a rate of $5\mathrm{ft/sec}$. Assume the scenario can be modeled with right triangles. At what rate is the length of the person's shadow changing when the person is 16 ft from the lamppost?

14. An observer stands 700 ft away from a launch pad to observe a rocket launch. The rocket blasts off and maintains a velocity of $900\mathrm{ft/sec}$. Assume the scenario can be modeled as a right triangle. How fast is the observer to rocket distance changing when the rocket is 2400 ft from the ground?