Calculus Honors – Notes 1: Review of Functions

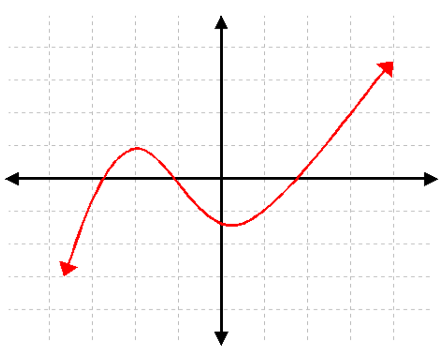
Relations and Functions

Relation
A relation is a set of ordered pairs (x, y). The relation can be described by any method that associates pairs between the values of the variable *y* and the variable *x*.

Function
A function *f* from set A to set B is a rule of correspondence that assigns to each element *x* in the set A **<u>exactly</u>** one element *y* in the set B. In other words, a function is a relation in which the input values (*x*) do not repeat in any ordered pairs.

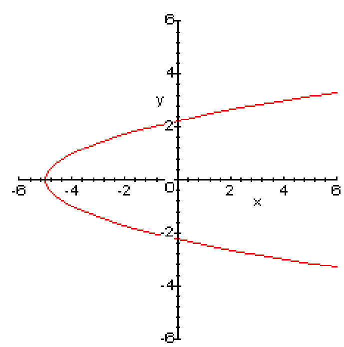
Vertical Line Test
If no vertical line intersects a given graph in more than one point, then the graph is the graph of a function.

Examples:



This represents y as a function of x.

No vertical line can pass through two points on this graph.



Here, y is NOT a function of x.

A vertical line can pass through Two points on this graph.

Domain

The **domain** is the set of all *x* values (or input values) of the function.

Range

The **range** is the set of all *y* values (or output values) of the function.

Secant Line

A line through any two points on a curve is called a **secant line**.

Difference Quotient

The slope formula is called the **difference quotient** and can be represented in different forms.

As the slope, m, of a secant line between two points $(a, f(a))$ and $(x, f(x))$, the difference quotient is:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

The difference quotient used in calculus considers the secant line between two points that are h units apart in x, $(x, f(x))$ and $(x + h, f(x + h))$, and is expressed as:

$$\frac{f(x + h) - f(x)}{h}$$

Zero of a Function

Any x value for which $f(x) = 0$ is called a **zero** of the function. Real zeros are also seen on the graph of the function as x-intercepts.

Function Notation

We describe the function has having an **input** and an **output**.

The input variable is usually x, although we can define any input variable we want.
The output variable is usually y, although again we can define any output variable we want.

The **function value** is the same as the **output value,** or the y **value.**

So, when we use function notation, it is another way of writing the output variable *y*.
We commonly call functions by letters. f is a commonly used letter to refer to functions.

If a relation between *y* and *x* is a function, then we can write:

$$y = f(x)$$

We say, "y equals f of x." Note that the parentheses here do NOT mean to multiply!

Then we can write the function rule with the function label in place of *y*.

NOTATION: **<u>Bounded</u> Intervals on the Real Number Line**

Interval	Interval Type	Inequality	Graph
$x \in [a, b]$	Closed		
$x \in (a, b)$	Open		
$x \in [a, b)$	Half-closed		
$x \in (a, b]$	Half-closed		

NOTATION: **<u>Unbounded</u> Intervals on the Real Number Line**

Interval	Interval Type	Inequality	Graph
$x \in [a, \infty)$	Half-closed		
$x \in (a, \infty)$	Open		
$x \in (-\infty, b]$	Half-closed		
$x \in (-\infty, b)$	Open		
$x \in (-\infty, \infty)$	Entire Real Number Line		

Evaluating Functions

When we **evaluate** a function, we find the **y-value** for a specified *x* value.

Examples: Evaluate each given function *f* at the indicated values:

1. $f(x) = 3x^3 - 5x^2 + 10x - 3;\ \ f(-1), f(-2)$ (Hint: use synthetic substitution!)

2. $f(x) = 4x^3 + 10x^2 + 19; f(-3), f(\sqrt{2})$

3. $f(x) = \sqrt{16 - x^2}; f(-4), f(\sqrt{3})$

We can also evaluate a function for a given expression, which may produce a real number value or an expression in terms of the variable input.

Examples: Evaluate each given function *f* for the indicated expression:

1. $f(x) = -3\sqrt{x-5} + 12; f(a), f(-a)$

2. $f(x) = -2x^3 - 5x; f(a+h), -f(a)$

Piecewise-Defined Functions

When a function has different rules for different values of the input variable, it can be defined **piecewise.**

Example:

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Evaluate this function when $x = -1$, $x = 0$, and $x = 1$. Use function notation.

Function Operations

Let *f* and *g* be functions. If the domain of both *f* and *g* is the set of all real numbers, then the following combinations also have the domain of the set of all real numbers.

1. **Sum** of f and g: $f + g = (f + g)(x) = f(x) + g(x)$

Example: If $f(x) = 3x + 3$ and $g(x) = -4x + 1$,
find $(f + g)(x)$ *and* $(f + g)(10)$

2. **Difference** of f and g: $f - g = (f - g)(x) = f(x) - g(x)$

Example: If $f(x) = 4x - 3$ and $g(x) = x^3 + 2x$, find $(f - g)(x)$ *and* $(f - g)(4)$

3. **Product** of f and g: $fg = (f \cdot g)(x) = f(x) \cdot g(x)$

Example: If $f(x) = x^2 + 2x + 4$ and $g(x) = -3x + 2$,
find $(f \cdot g)(x)$ and $(f \cdot g)(1)$

When we use division, we must prevent division by zero by excluding any *x* values that would result in division by zero from the domain of the quotient function.

4. **Quotient** of f and g: $f/g = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

Example: If $f(x) = 3x + 2$ and $g(x) = 2x - 4$, find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(3)$.

Also identify the domain of $\left(\frac{f}{g}\right)(x)$.

5. **Composition** of f and g: $f \circ g = (f \circ g)(x) = f(g(x))$

Given functions $f(x) = x + 2$ and $g(x) = 4 - x^2$, find

a) $f(g(x))$

b) $(g \circ f)(x)$

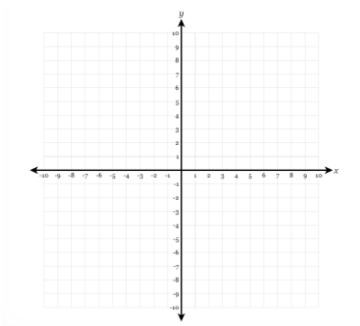
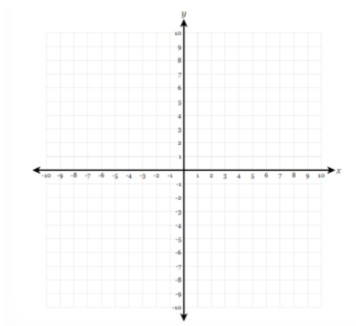
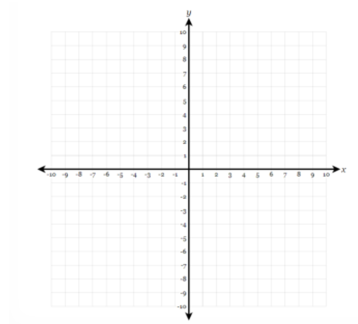
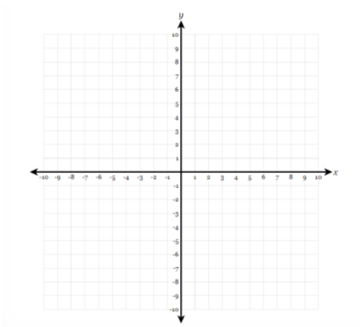
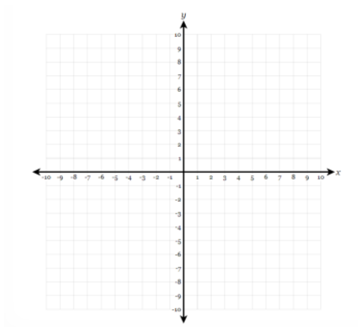
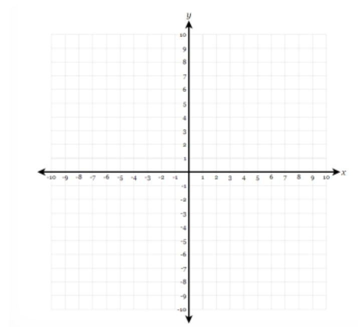
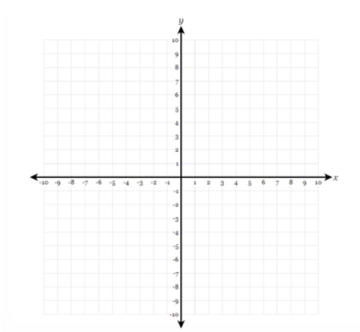
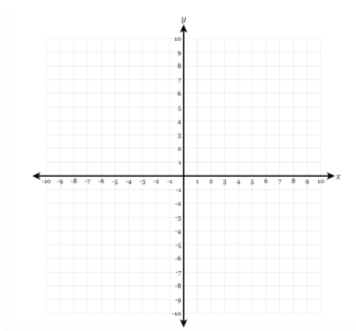
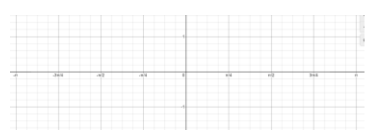
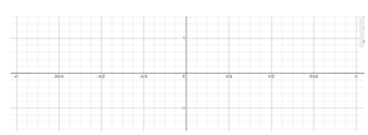
c) $g(f)(-2))$

d) $(g \circ f)(3c)$

Families of Functions

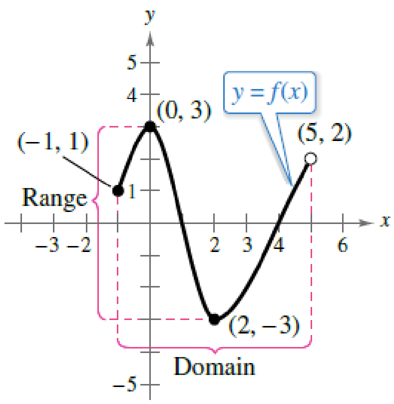
Function Type	General Form	Common Characteristics
Any Polynomial	$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$	Domain is all real numbers; function is smooth and continuous; end behaviors go to $\pm\infty$
Constant	$f(x) = c, c \neq 0, c \in \mathbb{R}$	
Linear	$f(x) = mx + b, m \neq 0, m, b \in \mathbb{R}$	
Quadratic	$f(x) = ax^2 + bx + c, a \neq 0,$ $a, b, c \in \mathbb{R}$	
Polynomial of Even Degree	$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0,$ n is even	
Polynomial of Odd Degree	$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0,$ n is odd	
Power	$f(x) = ax^n, a, n \in \mathbb{R}$	
Absolute Value	$f(x) = \lvert variable\ expression \rvert$	
Exponential	$f(x) = a \cdot b^x,$ $a, b\ \neq 0, b > 0, b \neq 1, a, b \in \mathbb{R}$	
Logarithmic	$f(x) = a \cdot ln(u)$ or $f(x) = a \cdot log_b(u),$ $a, b\ \neq 0, b > 0, b \neq 1, a, b \in \mathbb{R},$ u is a variable expression >0	
Trigonometric	$f(x)$ contains $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\csc(x)$, or $\cot(x)$	
Rational	$f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0,$ $p(x), q(x)$ are polynomial	

Graphs of Some Commonly Used Parent Functions

$f(x) = x$ 	$f(x) = x^2$ 	$f(x) = x^3$ 
$f(x) = \|x\|$ 	$f(x) = \sqrt{x}$ 	$f(x) = \frac{1}{x}$ 
$f(x) = e^x$ 	$f(x) = \ln(x)$ 	$f(x) = \sin(x)$  $f(x) = \cos(x)$ 

✋ *SKILL:* Finding the Domain and Range of a Function

Use the graph of the function to describe the domain and the range.



Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval when, for any x_1 and x_2 in the interval,

$x_1 < x_2$ implies

A function f is **decreasing** on an interval when, for any x_1 and x_2 in the interval,

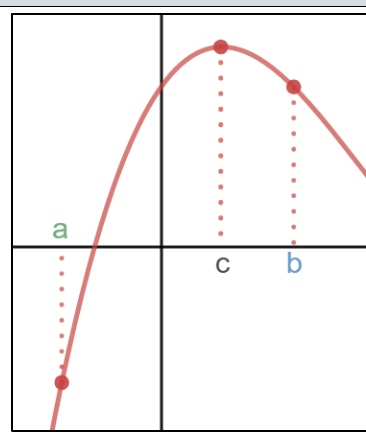
$x_1 < x_2$ implies

A function f is **constant** on an interval when, for any x_1 and x_2 in the interval,

Note that these definitions require us to analyze the graph from **left to right**.

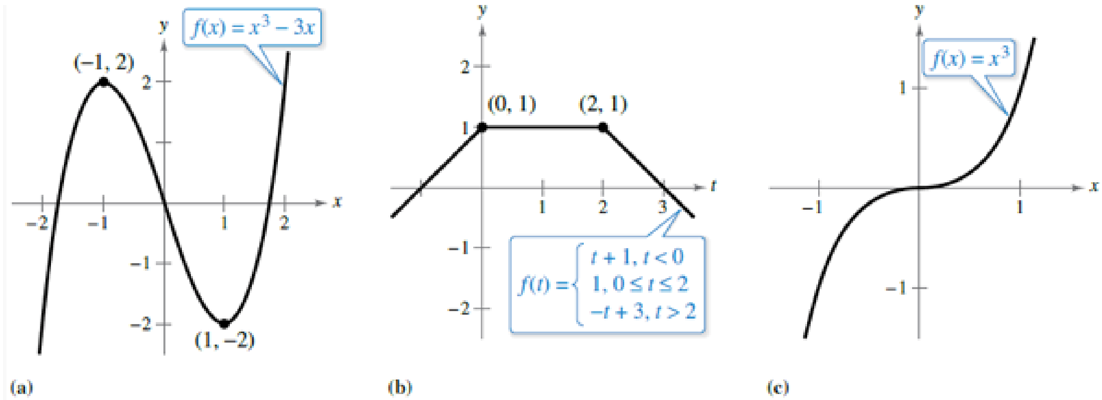
✋ *SKILL:* Describing Function Behavior

The graph of a function is shown to the right.
At $x = a$, $x = b$, and $x = c$ state whether y is increasing, decreasing, or neither as *x* increases.
then state whether the rate of change is fast or slow.



SKILL: Describing Function Behavior

Determine the open intervals on which each function is increasing, decreasing, or constant.



DEFINITIONS: Local Minimum and Local Maximum

Suppose c is an interior point of some interval I on which f is defined. If f(c) ≥ f(x) for all x in I, then f(c) is a **local maximum** value of f.

If f(c) < f(x) for all x in I, then f(c) is a **local minimum** value of f.

So, what this really means is, there is a point $(c, f(c))$ that sits at the bottom of a "dip" (local minimum) or at the top of a "peak" (local maximum).

Note: Local minima and maxima cannot occur at an ______________________
____.

✋ *SKILL:* Approximating the Local Minimum or Local Maximum

Use your calculator "minimum" or "maximum" feature to approximate the local minimum or maximum of the function.

$f(x) = 3x^5 - 5x^3 - 4x$

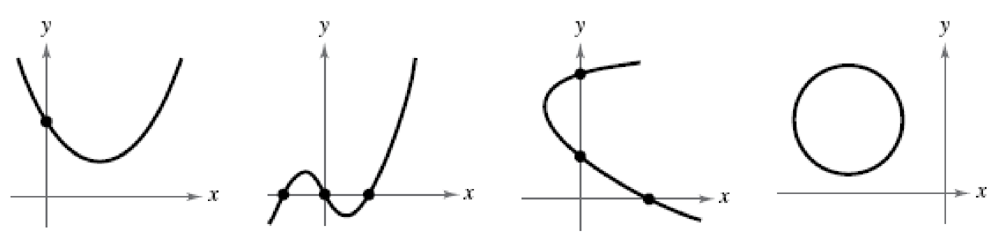
✍ *DEFINITIONS*: Intercepts of a Graph

Solution points of an equation that include a zero as one (or both) of the coordinates are called **intercepts.**

An ***x*-intercept** has a ____________________ of zero.

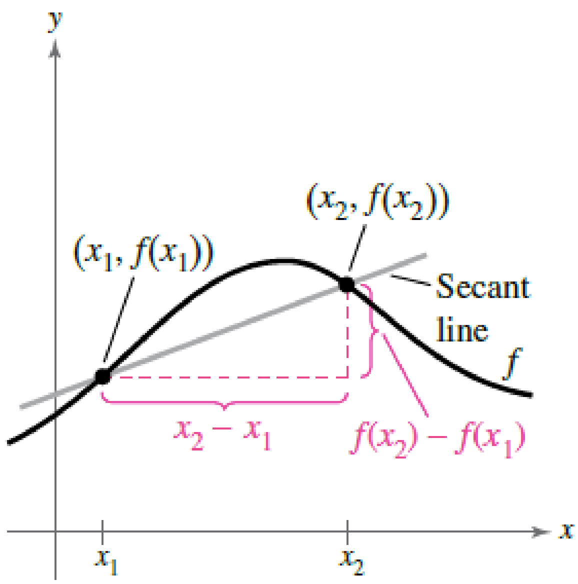
A ***y*-intercept** has an ____________________ of zero.

It is possible for graphs to have no intercepts, one intercept, or more than one intercept.



✋ *Application:* Average Rate of Change – Slope of a Secant Line

For a nonlinear graph, the **average rate of change** between any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the slope of the line through the two points. The line through the two points is called a **secant line,** and the slope of this line is denoted as m_{sec}.



✋ *SKILL:* Finding Average Rate of Change of a Function

Find the average rate of change of $f(x) = x^3 - 3x$ from

a) $x_1 = -2$ to $x_2 = -1$

b) $x_1 = 0$ to $x_2 = 1$

✋ *SKILL:* **Finding Average Speed**

When a function models displacement (distance) as a function of time, the average rate of change of the function can describe the speed of the motion.

Example:

The distance (*s* in feet) a moving car is from a stoplight is given by the function:

$$s(t) = 20t^{2/3}$$

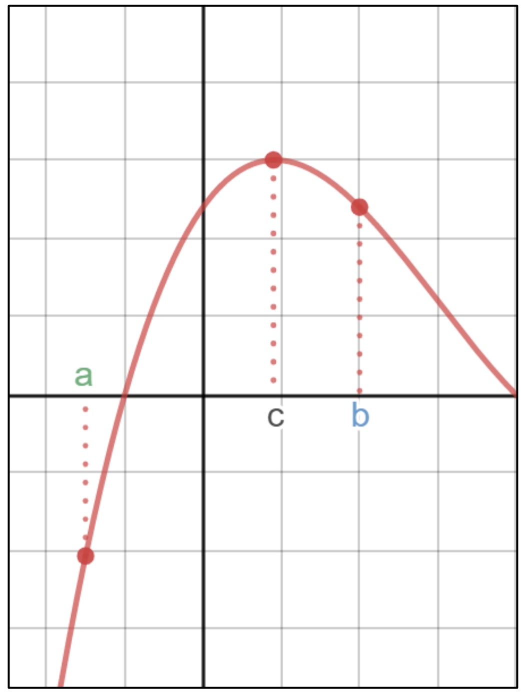
Where *t* is the time (in seconds). Find the average speed of the car

a) from $t_1 = 0$ to $t_2 = 4$

b) from $t_1 = 4$ to $t_2 = 9$

Rate of Change by Graph

Given part of the graph of function f at right, approximate the average rate of change of f from



a) $x = a$ to $x = b$.

b) $x = a$ to $x = c$.

c) $x = c$ to $x = b$.