Intro to Calculus
Limits and Graphing Rational Functions

Finding Limits of Functions

So far we know how to find limits of sequences, now we will extend that concept and look at limits of functions. Since functions can use any real number (sequences were only allowed to use 1, 2, 3, … etc) we need to look at 3 types of limits.

$$\lim_{x\to\infty} f(x) \qquad \lim_{x\to-\infty} f(x) \qquad \lim_{x\to c} f(x)$$

where c is a real number

A limit is the ***y-value*** the function is approaching at a given x-value.

Limits to Infinite

We do these limits almost exactly the same way we did limits for sequences.

1. If the degree of the numerator is less than the degree of the denominator, lim = 0

2. If the degree of the numerator is greater than the degree of the denominator, lim = DNE.

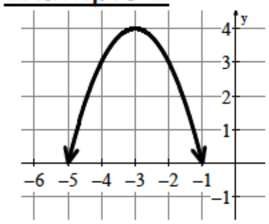
Here is one difference from sequences, when you head towards - ¥ you might get:

3. If the degree of the numerator = degree of denominator, the limit is the ratio of the lead coefficients.

What is a limit?

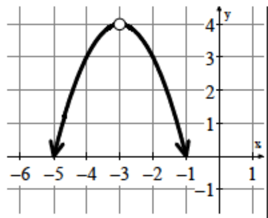
A limit is the ___________ a function _______________ from both the left and the right side of a given ____________.

Example 1



$f(-3) =$

$\lim_{x\to -3} f(x) =$



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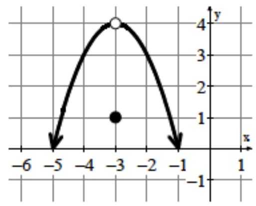

$f(-3) =$

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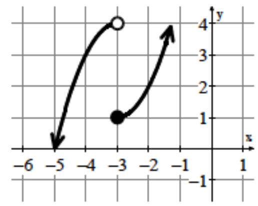
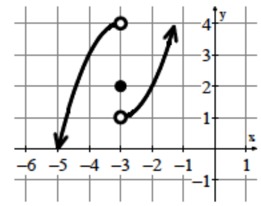


$f(-3) =$

$\lim_{x\to -3} f(x) =$



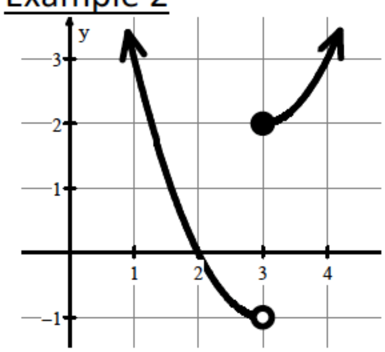
$f(-3) =$

$\lim_{x\to -3} f(x) =$

What is a one-sided limit?

A one-sided limit is the ____________ a function approaches as you approach a given ____________ from either the _______ or _______ side.

Example 2



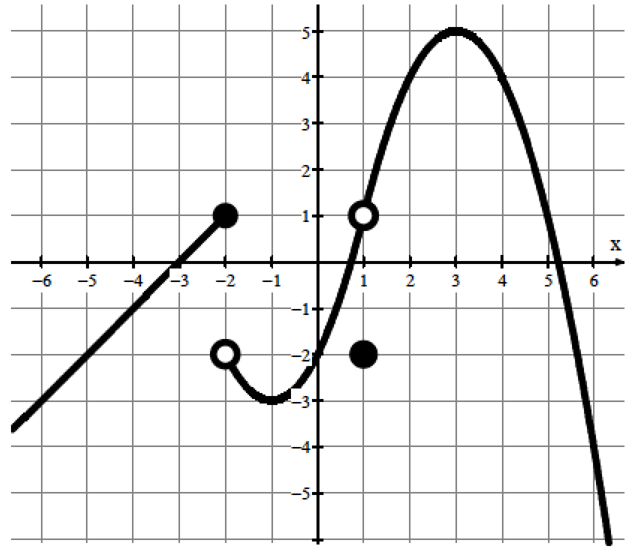
"The limit of f as x approaches 3 from the left side is −1." $\lim_{x\to 3^-} f(x) = -1$

"The limit of f as x approaches 3 from the right side is 2." $\lim_{x\to 3^+} f(x) = 2$

Example 3

a. $\lim_{x\to -2^-} f(x) =$	b. $\lim_{x\to -2^+} f(x) =$	c. $\lim_{x\to -2} f(x) =$
d. $\lim_{x\to 1} f(x) =$	e. $\lim_{x\to 0} f(x) =$	f. $\lim_{x\to 3^-} f(x) =$
g. $\lim_{x\to -1} f(x) =$	h. $\lim_{x\to -3} f(x) =$	i. $f(-2) =$

j. $f(1) =$



When does a limit not exist?

1.

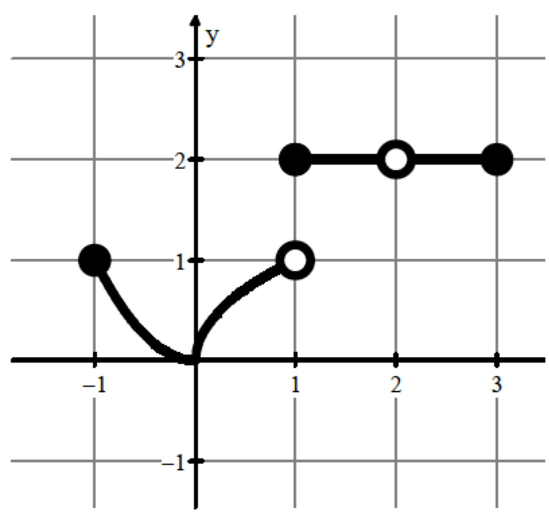
2.

3.

Example 5

Write **T** (true) or **F** (false) under each statement.
Use the graph on the right.

a. $\lim_{x\to -1^+} f(x) = 1$	b. $\lim_{x\to 2} f(x) = 2$	c. $\lim_{x\to 1^-} f(x) = 1$
d. $\lim_{x\to 1^+} f(x) = 2$	e. $\lim_{x\to 1} f(x) =$ does not exist	
f. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$	g. $\lim_{x\to 2} f(x) =$ does not exist	



Limits #1

Find the following limits. Use your GC to graph the function if needed.

1. $\lim_{x\to-\infty}\dfrac{x+2}{x^2+x+1}$

2. $\lim_{x\to-\infty}\dfrac{3x^3}{3x^2-2}$

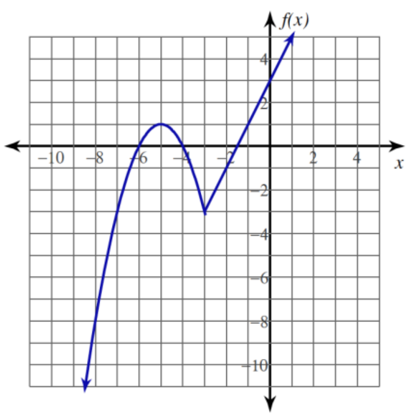
3. $\lim_{x\to-\infty}\dfrac{2x^2}{x^2-4}$

4. $\lim_{x\to\infty}\dfrac{3x^2}{4x+4}$

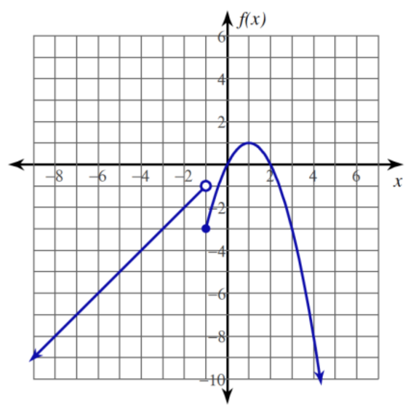
5. $\lim_{x\to\infty}x^3 - 4x^2+5$

6. $\lim_{x\to-\infty}\dfrac{\sqrt{2x^2+1}}{4x+2}$

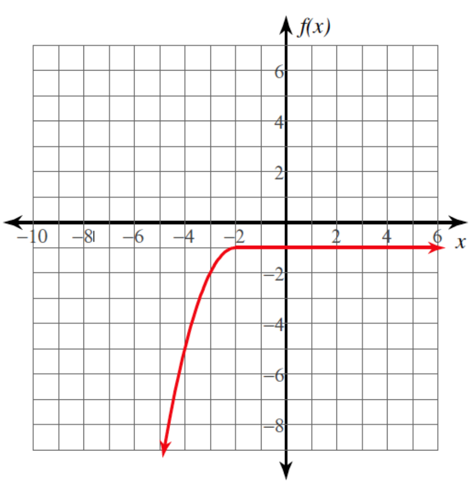
7. $\lim_{x\to-3} f(x),\ f(x)=\begin{cases}-x^2-10x-24, & x\le -3\\ 2x+3, & x>-3\end{cases}$



8. $\lim_{x\to-1} f(x),\ f(x)=\begin{cases}x, & x<-1\\ -x^2+2x, & x\ge -1\end{cases}$



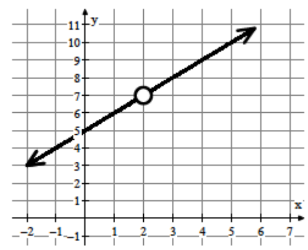
9. $\lim_{x\to-2} f(x),\ f(x)=\begin{cases}-x^2-4x-5, & x\le -2\\ -1, & x>-2\end{cases}$



10. $\lim_{x\to-1}\dfrac{3|x+1|}{x+1}$

Limits Analytically

Recall: Find the limit graphically. $\lim_{x\to 2}\dfrac{x^2+3x-10}{x-2}$



A limit is …

<u>**Steps to Finding Limits Analytically**</u>

1.

2.

a.

b.

<u>**Direct Substitution:**</u>

1. $\lim_{x\to -1}\left(x^2+2x-4\right)$

2. $\lim_{x\to 2}\sqrt{3x-2}$

3. $\lim_{x\to 4}5$

<u>**Factor and Cancel:**</u>

4. $\lim_{x\to 0}\dfrac{4x^2-5x}{x}$

5. $\lim_{x\to -7}\dfrac{2x^2+13x-7}{x+7}$

6. $\lim_{x\to -2}\dfrac{3x^3-6x^2-24x}{3x^2+6x}$

Rationalize:

7. $\lim_{x \to 0} \frac{\sqrt{x+5} - \sqrt{5}}{x}$

8. $\lim_{x \to 5} \frac{\sqrt{x+4} - 3}{x - 5}$

Two Variables:

9. $\lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$

10. $\lim_{h \to 0} \frac{2(x+h)^2 + 5(x+h) - 1 - (2x^2 + 5x - 1)}{h}$

Limits #2

Find the value of each limit on **separate** paper. Do these without a graphing calculator.

1. $\lim_{x\to -2}\left(3x^2 - x + 1\right)$

2. $\lim_{x\to 1} 3$

3. $\lim_{x\to 5}\sqrt{4x - 9}$

4. $\lim_{x\to 0}\frac{x^2 + 2x - 8}{x - 4}$

5. $\lim_{x\to 5}(x+1)^2$

6. $\lim_{x\to -2}\frac{x^2 - 4x - 10}{x}$

7. $\lim_{x\to -7}\frac{2x^3 + 11x^2 - 21x}{x^2 + 7x}$

8. $\lim_{x\to 0}\frac{\sqrt{x+7} - \sqrt{7}}{x}$

9. $\lim_{x\to 1}\frac{\sqrt{x+5} + \sqrt{6}}{x}$

10. $\lim_{x\to -1}\sqrt{3 - x}$

11. $\lim_{x\to 2}\frac{\sqrt{5x - 6}}{x}$

12. $\lim_{x\to 2}\left(x - x^2\right)$

13. $\lim_{x\to 0}\left(-2\right)$

14. $\lim_{x\to 8}\frac{x^2 + 2x - 80}{x - 8}$

15. $\lim_{x\to 4}\frac{5x^2 - 21x + 4}{x - 4}$

16. $\lim_{x\to \frac{1}{2}}\frac{2x^2 + x - 1}{2x - 1}$

17. $\lim_{x\to 0}\frac{3x^2 + 5x}{x}$

18. $\lim_{h\to 0}\frac{6 - 3(x+h) - (6 - 3x)}{h}$

19. $\lim_{h\to 0}\frac{(x+h)^2 + 6(x+h) - \left(x^2 + 6x\right)}{h}$

20. $\lim_{x\to 7}\frac{\sqrt{x+9} - 4}{x - 7}$

21. $\lim_{x\to 0}\frac{\sqrt{x+9} - 3}{x}$

22. $\lim_{h\to 0}\frac{4(x+h)^2 - 5(x+h) - 2 - \left(4x^2 - 5x - 2\right)}{h}$

23. $\lim_{x\to 5}\frac{2x^2 - 17x + 35}{5 - x}$

24. $\lim_{x\to -3} 14$

Continuity

Continuous Functions

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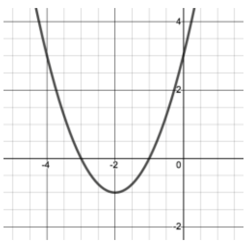
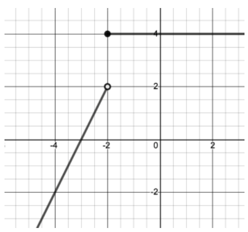
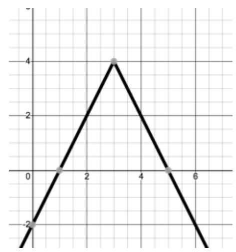
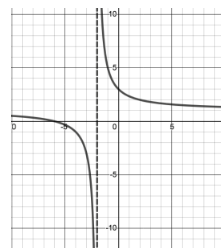
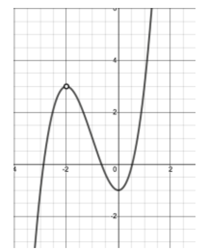
Are these functions continuous or discontinuous?

______________ ______________ ______________ ______________ ______________

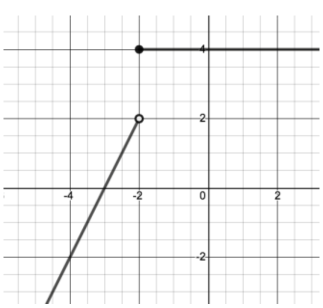
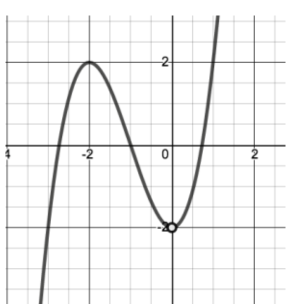
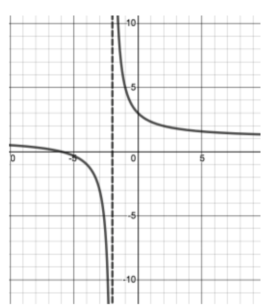
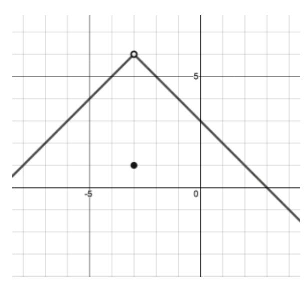
There are 3 conditions for continuity. A function f(x) is continuous at a real number c if:

1. ______________________________________

2. ______________________________________

3. ______________________________________

State why each function is discontinuous and state the x-value where the function is discontinuous.

Rational Functions

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Examples:

1. $f(x) = \dfrac{x+3}{x-7}$

2. $f(x) = \dfrac{x^2+4x}{(x-3)(x+1)}$

3. $f(x) = \dfrac{2x^2-x+5}{x^2+4x-12}$

Are the following functions continuous or discontinuous? If it is discontinuous, state the x-value where the function is discontinuous.

1. $f(x) = \dfrac{3x+9}{x^2-4}$

2. $f(x) = 3x^4 - 7x^3 + x^2 - 5x + 31$

3. $f(x) = \dfrac{-2x}{2x^2-9x-5}$

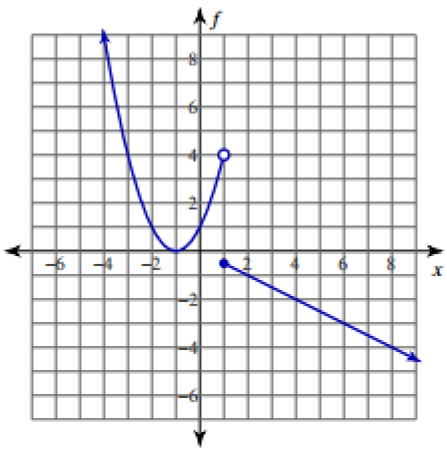
4. $f(x) = \dfrac{|3x-8|}{25}$

Limits #3

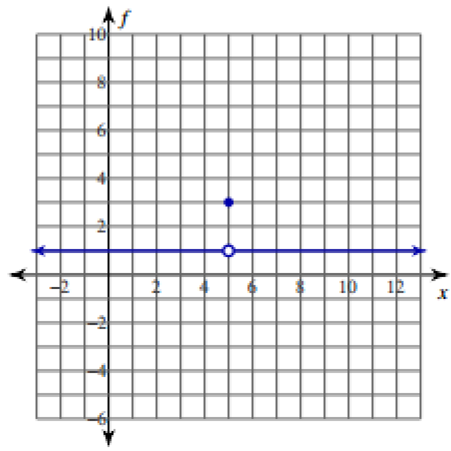
Continuous Functions

State if the following functions are continuous or not. If it is discontinuous, state the x-value(s) where it is discontinuous.

1. $f(x) = \begin{cases} x^2 + 2x + 1 & x < 1 \\ -\frac{x}{2} & x \geq 1 \end{cases}$



2. $f(x) = \begin{cases} 1 & x \neq 5 \\ 3 & x = 5 \end{cases}$



3. $f(x) = \frac{x^2 - x - 2}{x + 1}$

4. $f(x) = \frac{x^2}{2x + 4}$

5. $f(x) = x^2 - x - 12$

Determine if each function is continuous. If it isn't, state x-location of the discontinuity.

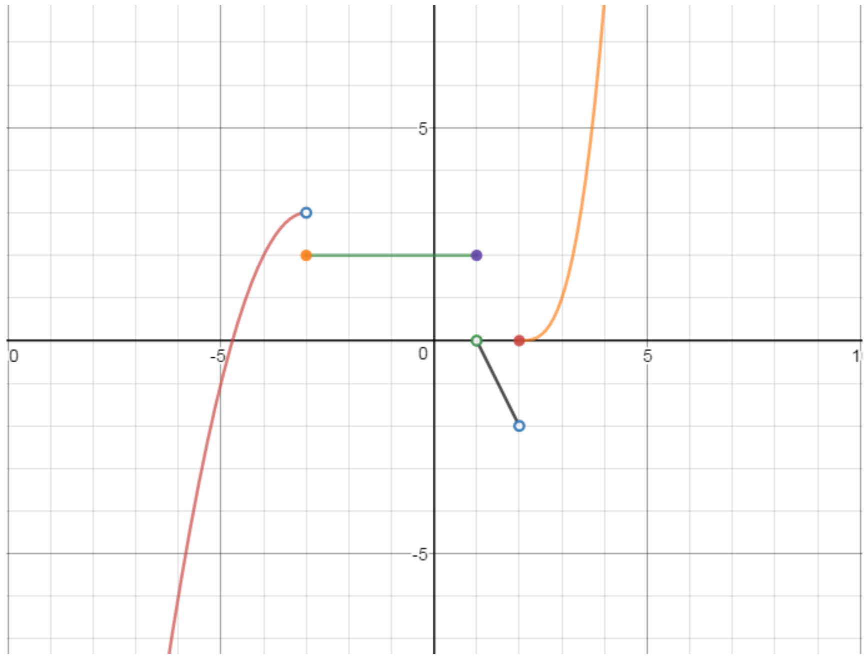
6. $f(x) = -\frac{x^2}{2x - 4}$

7. $f(x) = \frac{x + 1}{x^2 - x - 2}$

8. $f(x) = \frac{x^2 + x - 6}{x^2 + 3x + 2}$

9. $f(x) = \left| \frac{3x + 2}{4} \right|$

For questions 10 – 18, use the graph below.



10. $\lim_{x\to\infty} f(x)$

11. $\lim_{x\to -3^-} f(x)$

12. $\lim_{x\to 3^+} f(x)$

13. $\lim_{x\to 3} f(x)$

14. $\lim_{x\to 2^+} f(x)$

15. $\lim_{x\to 2^-} f(x)$

16. $\lim_{x\to -1} f(x)$

17. $\lim_{x\to -\infty} f(x)$

18. f(-3)

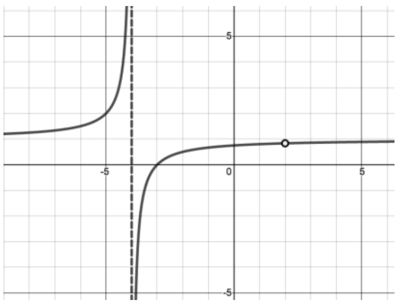
19. Sketch an example of a function with discontinuities at x = 1, 2, and 3.

Types of Discontinuities

Removable – ____

Non-removable – ____

A Rational Function can have:



Removable Discontinuities

Non-removable

State where the following functions are discontinuous and classify the type of discontinuity.

1. $f(x)=\dfrac{2x^2-7x-15}{x-5}$

2. $f(x)=\dfrac{x^2-1}{2x^2+13x-7}$

3. $f(x)=\dfrac{x^2-x-12}{x^2+3x-28}$

Finding the Hole

Finding the vertical asymptote

Example 1: $f(x)=\dfrac{x^2+4x+3}{x^2-x-2}$

Example 2: $f(x)=\dfrac{2-x}{4x-x^3}$

Example 3: $f(x)=\dfrac{x+3}{(x+3)(x^2+6x+9)}$

Example 4: $f(x)=\dfrac{x}{x^2+13}$

Other Types of Asymptotes

HORIZONTAL ASYMPTOTES ARE NOT DISCONTINUITIES OF THE FUNCTION!!!

Finding Horizontal Asymptotes

State the equation of the horizontal asymptotes.

1. $f(x)=\dfrac{(x+1)}{(x+3)(x-5)}$

2. $f(x)=\dfrac{4x^3-5x^2}{2x^3+3x}$

3. $f(x)=\dfrac{x^2-3x-10}{x-4}$

Limits #4
Try These.
For each of the following:
a. Identify the x-value of the discontinuities, if there are any.
b. Identify whether the discontinuities are removable or non-removable.
c. Find the coordinates of any holes and the equations of any vertical asymptotes.
d. State the equation of the horizontal asymptote, if there is one.

a. $f(x) = \dfrac{2x - x^2}{2 - x}$

b. $f(x) = \dfrac{x^2 + 4x + 3}{x^2 - x - 2}$

c. $f(x) = \dfrac{x^2 - 25}{x^2 - 4x - 5}$

d. $f(x) = \dfrac{x}{x + 1}$

e. $f(x) = \dfrac{x^2 - 4x + 4}{x - 2}$

f. $f(x) = \dfrac{2x - 5}{x^2 + 16}$

g. $f(x) = \dfrac{x + 3}{(x - 3)^2}$

h. $f(x) = \dfrac{x + 5}{x^2 + 10x + 25}$

i. $f(x) = \dfrac{2}{x^2 - 2x}$

j. $f(x) = \dfrac{2x + 3}{x^2 - 4}$

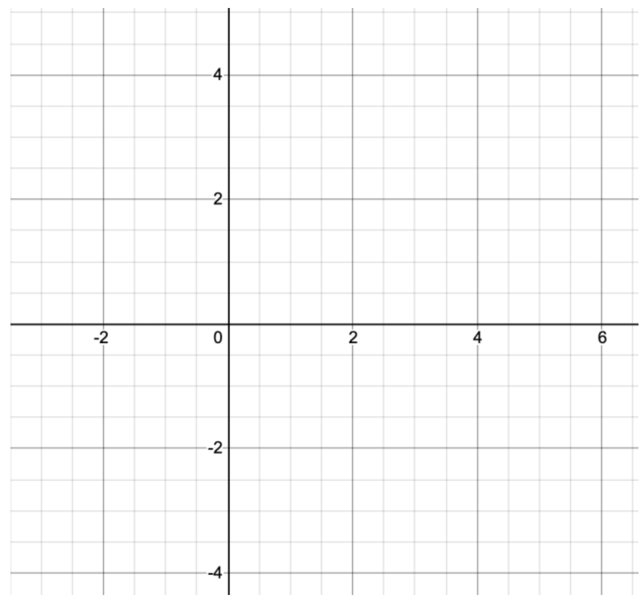
Graphing Rational Functions.

1. Locate the discontinuities. Decide whether they are removable or non-removable.

a. Find the equation of any vertical asymptote(s).

b. Find the coordinates of any hole(s).

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3}$$



2. Find the horizontal asymptotes by considering $\lim_{x \to \infty} \frac{f(x)}{g(x)}$.

$$\lim_{x \to \infty} \frac{x^2 - 9}{x^2 - 2x - 3}$$

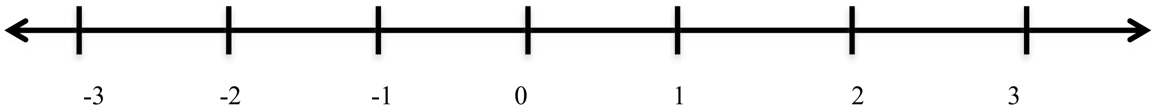
3. Find the x-intercept(s). Locate the zeros of the top of the reduced function.

$$f(x) = \frac{x+3}{x+1} \quad .$$

4. Find the y-intercept by plugging zero in for x. Put this into the graph.

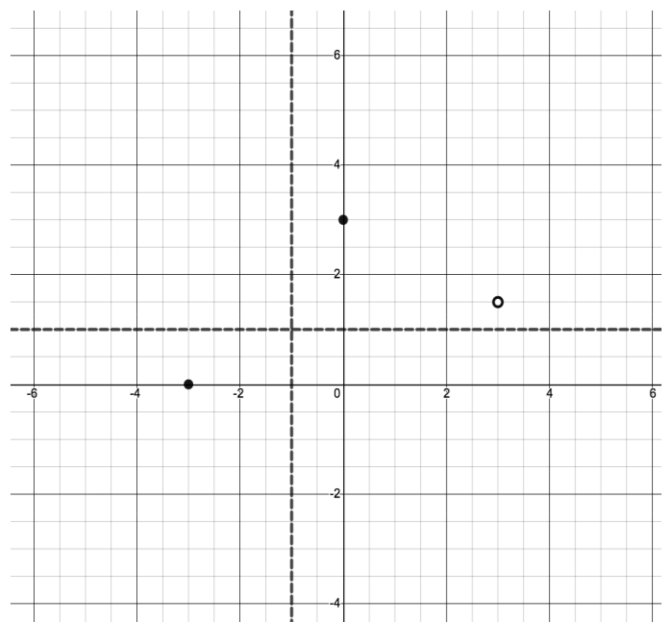
$$f(0) = \frac{x+3}{x+1}$$

5. Determine the intervals where the function is positive and negative. We will use a number line to do this, think back to inequalities. Plot all your zeros on the number line and then check each region using the reduced function.



6. In the positive regions the function will be above the x-axis and in the negative regions the function will be below the x-axis.

7. If x = c is a vertical asymptote, then the graph will approach ∞ in positive regions and $-\infty$ in negative regions.



8. Try to draw your graph.

9. If necessary, create a table of values to find a few more points on the graph.

x	y
-4	
-2	
1	
4	
5	

Example 2: $f(x) = \dfrac{2x+3}{x^2 - 4}$

Limits #5

Try These. Do all work on separate paper.

For each of the following:

a. State where the graph is discontinuous and what type of discontinuity.
b. Find the equations of any vertical asymptotes.
c. Find the coordinates of any holes.
d. Find any x-intercepts.
e. Find the y-intercept.
f. Find the equation for the horizontal asymptote, if there is one.
g. Use a number line to see where the function is positive and negative.
h. Create a table of values if you need more points. (This is optional)
i. Graph the function, make sure to show b – f on your graph.

1. $f(x)=\frac{4}{x^2-3x}$

2. $f(x)=\frac{x-4}{4x+16}$

3. $f(x)=\frac{x^2+x-6}{x^2+4x+3}$

4. $f(x)=\frac{x+4}{-2x-6}$

5. $f(x)=\frac{2x^2+10x+12}{x^2+3x+2}$

Limits #6
More Practice. Do all work on separate paper.

For each of the following:

a. State where the graph is discontinuous and what type of discontinuity.
b. Find the equations of any vertical asymptotes.
c. Find the coordinates of any holes.
d. Find any x-intercepts
e. Find any y-intercepts
f. Find the equation for any horizontal asymptotes.
g. Use a number line to see where the function is positive and negative
h. Create a table of values if you need more points.
i. Graph the function, make sure to show b – f on your graph.

1. $f(x) = \dfrac{(2x - 3)(x + 1)}{(x + 1)(x - 4)}$

2. $f(x) = \dfrac{x - 1}{x^2}$

3. $f(x) = \dfrac{x}{x^2 - 4}$

4. $f(x) = \dfrac{10}{x^2 + 2}$

5. $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}$